

Exact Growth of Entanglement and Dynamical Phase Transition in Global Fermionic Quench

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Abstract

Critical quantum quench of free Dirac fermions in an infinite system is examined carefully. A much broader analysis, with more emphasis on free scalar fields, has been done in [1]. For specially prepared squeezed states of the massive theory, quenched states obtained are Calabrese-Cardy(CC) states and generalized Calabrese-Cardy(gCC) states with higher-spin charges. Exact time dependence of correlators are computed showing thermalization explicitly. We also calculate the exact monotonic growth of entanglement entropy in CC states. In case of gCC states, for a particular charge, we show that there is a dynamical phase transition from monotonic to non-monotonic entanglement entropy growth when the effective chemical potential is increased beyond a critical value.

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1 Introduction and Summary

Thermalization in unitary quantum field theories has been a topic of great significance. Using AdS/CFT correspondence, it has also been linked to black hole formation [2, 3]. One of the current views of thermalization is that of the thermalization of a finite subsystem, in which the conjugate subsystem is considered the heat bath. In other words, it is the thermalization observed by an observer who has access to only a subsystem of the full system. It can also be considered as if the ‘fine-grained’ observables¹ are spatially widely separated bilocal or higher point observables. Starting from a pure state, in the high energy (high effective

¹Observables which show the non-thermal behaviour of the pure state, in contrast to ‘coarse-grained’ observables which cannot distinguish between the pure state and the thermal ensemble.

temperature) limit, the final thermal entropy observed by such an observer is actually the entanglement entropy of the subsystem with its conjugate. Obviously, the pure state has to be a time-dependent state. Closely related to thermalization (equilibration in general), the study of time-dependent states after a quantum quench has also been of great interest [4, 5]. Quantum quench is the process in which the parameters of the Hamiltonian of a system in a certain state are changed with time. After the quantum quench, in the long time limit, if the subsystem of our interest looks like a thermal ensemble, in the sense that the expectation values of observables in the finite subsystem have the same expectation values as in a thermal ensemble, then we say that the system has thermalized. In this paper, we will be mainly considering quantum quench as the preparation of the time-dependent states of our interest.

We will also restrict ourselves to critical quantum quenches, in which the final Hamiltonian is a critical Hamiltonian, i.e., the corresponding theory is a conformal field theory (CFT). More specifically, we will be considering free fermions in which starting from a certain state in the massive theory, the mass is set to zero gradually or suddenly. In more general theories, starting from the ground state of a gapped theory, it has been proposed [6, 7] that the state obtained after the critical quench is a Calabrese-Cardy (CC) state which has the form $e^{-\kappa_2 H} |B\rangle$, where κ_2 is a scale given by the initial gap and the other scales of the quench process, H is the Hamiltonian of the CFT and $|B\rangle$ is a conformally invariant boundary state. It has been shown that such a state thermalizes to a thermal ensemble with temperature $T = 1/\beta = 1/(4\kappa_2)$. This result has also been generalised to the case in which the final theory has other conserved charges of local currents [8]. The corresponding ansatz for the state after quench from ground state is a generalized Calabrese-Cardy (gCC) states which have the form $e^{-\kappa_2 H - \kappa_4 W_4 - \kappa_6 W_6 - \dots} |D\rangle$ where again the parameters $\kappa_2, \kappa_4, \kappa_6, \dots$ are given by the initial gap and other scales in the quench process, e.g. $\delta t = 1/\rho$ the time taken to set the mass to zero, and W_4, W_6, \dots are the conserved charges of local currents. In this case also, it has been shown that the state thermalizes into a generalized Gibb's Ensemble (GGE) with the density matrix $e^{-\beta H - \mu_4 W_4 - \mu_6 W_6 - \dots}$ where the corresponding temperature and chemical potentials are $T = 1/\beta = 1/(4\kappa_2)$, $\mu_4 = 4\kappa_4$, $\mu_6 = 4\kappa_6, \dots$.

The gCC state ansatz has been shown to be true for mass quenches in free scalar and free fermion theories in a recent paper (MPS) [1]. Starting from the ground state of the massive theories, the quenched states obtained are of the gCC form with infinite number of charges W_{2n} with $n \in \mathbb{N}$ ($W_2 = H$). For the scalar theory, it was also found that naively taking the sudden limit when the mass profile is taken to be a step function, the final state is non-normalizable. For massless free scalar theory, $W_{2n} = \sum |k|^{2n-1} d_k^\dagger d_k$, where d_k^\dagger and d_k are the bosonic annihilation and creation operators.² It was also shown that starting from specially prepared squeezed states of the massive scalar theory, CC state and gCC state with finite number of charges can also be created. By calculating correlators, thermalization of these states were explicitly shown.

In this paper, we find similar results for the fermionic mass quench. In the sudden limit, starting from the ground state, we observe that the final state has divergent energy density, W_4, W_6, \dots . Again, as in the case of scalar fields in MPS, starting from specially prepared squeezed states using the sudden quench limit, we can prepare CC state and gCC state with a finite number of charges of our choice. For the CC state and the gCC state with finite

²The normalization of the charges differ from the normalization in [9, 10].

number of charges, we calculate correlators and explicitly show thermalization to thermal ensemble and GGE respectively.

Among the other calculable quantities, entanglement entropy (EE) is the most interesting one. The EE growth has been calculated (mostly numerically) in many dynamical systems, see for e.g. [6, 11, 12, 13, 14, 15, 16]. It has also been extensively examined in holographic systems [17, 18, 19, 20, 21]. Recently, non-monotonic EE growth consisting of an initial dip around the quench time has also been observed in a holographic set-up in [22].

Since our final theory consists of only massless Dirac fermions, so using bosonization, we could calculate EE in some of our time-dependent states. We are interested in EE of a single interval only. For CC states, we find that EE grows monotonically. The asymptotic time limit is given by the well-known expression from CFT in a thermal ensemble, $S_A = \frac{c}{3} \log(\sinh(\frac{\pi r}{\beta}))$, where for Dirac fermions $c = 1$ and the effective temperature $1/\beta = 1/4\kappa$. In case of gCC states, we are not able to calculate EE with the charges of the usual fermionic bilinear $\mathcal{W}_{1+\infty}$ currents. But we are able to calculate the EE with the fermionic charge corresponding to the bosonic charges $W_{2n} = \sum |k|^{2n-1} d_k^\dagger d_k$. These are the charges of bosonic bilinear \mathcal{W}_{2n} currents for $n > 1$. For such gCC states with the W_4 charge, we found a dynamical phase transition in which EE grows non-monotonically when the effective chemical potential μ_4 is greater than a critical value. Below this critical value, the EE growth is strictly monotonic.

In summary, the key results of the present work are:

1. For ground state quench, similar to the scalar quench, a naive sudden quench limit gives divergent conserved charges. Calculation of the correlators show equilibration explicitly. But the long distance and time and ultimately the stationary limit is significantly different from thermalization to a thermal ensemble. This is the same manifestation of the UV/IR mixing found in MPS.
2. Starting from appropriately prepared squeezed states of the massive theory, we can prepare CC and gCC states with specific W_{2n} charges using quench. Calculation of correlators in CC state and gCC states explicitly show thermalization to thermal ensemble and GGE respectively. Here again, for gCC state, the long time and long distance limit of the correlators have significant dependence on the chemical potentials. This is again another avatar of the UV/IR mixing.
3. For CC state, we are able to calculate the growth of entanglement entropy of a single interval explicitly in analytic form. The EE growth is strictly monotonically increasing for CC state. The stationary limit is, as expected, the entanglement entropy of a single interval in thermal ensemble.
4. We also calculate the EE growth of a single interval in gCC state with W_4 charge of the \mathcal{W}_{2n} representation of fermion corresponding to the \mathcal{W}_{2n} bilinear bosonic representation. We find dynamical phase transition in which the EE growth is monotonically increasing upto a critical value of κ_4 . Beyond the critical value, the EE growth is non-monotonic.

The outline of the paper is as follows:

In section 2, we solve the Dirac equation with time-dependent mass and from explicit solutions for a specific mass profile, we calculate the Bogoliubov coefficients for the transformation between the massive and massless modes. In section 3, we find the final state after the quench starting from the ground state and a few squeezed states of our interest. In sections 4 and 5, we calculate energy density and some correlators in the different quenched states that we obtained. The EE growth of a single subsystem in CC state is explicitly calculated in section 6. In section 7, we show the dynamical phase transition in the EE growth of a subsystem in a particular gCC state. Section 8 contains some discussions. The appendix contains details that we have omitted in the main sections.

2 Free Dirac fermions with time-dependent mass

The action for Dirac fermions with time-dependent mass is

$$S = - \int dx^2 [i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m(t)\bar{\Psi}\Psi] \quad (1)$$

The equation of motion (EOM) is

$$[i\gamma^0\partial_t - i\gamma^1\partial_x - m(t)]\Psi(x, t) = 0 \quad (2)$$

and we are interested in the solvable mass profile[23, 24]

$$m(t) = m[1 - \tanh(\rho t)]/2 \quad (3)$$

m is the initial mass and ρ is the only scale of the quench process. $\rho \rightarrow \infty$ is the sudden limit in which the mass is set to zero suddenly - much faster than any other length scale in the theory. It is easier to solve (2) in the Dirac basis in which γ_0 is diagonal. Since the system is translation invariant in the spatial x -direction, the solution ansatz is

$$\Psi(x, t) = [\gamma^0\partial_t - \gamma^1\partial_x - im(t)] e^{\pm ikx}\Phi(t) \quad (4)$$

Substitution in the EOM gives,

$$[\partial_t^2 + k^2 + m(t)^2 - i\gamma^0\dot{m}(t)] e^{\pm ikx}\Phi(t) = 0$$

where $\dot{m}(t) = \partial_t m(t)$.

$\Phi(t)$ is solved in the eigenbasis of γ^0 . For the two eigenvalues of γ^0 (1 and -1), the two solutions $\phi_+(t)$ and $\phi_-(t)$ are given by,

$$\begin{aligned} [\partial_t^2 + k^2 + m(t)^2 - i\dot{m}(t)] \phi_+(t) &= 0 \\ [\partial_t^2 + k^2 + m(t)^2 + i\dot{m}(t)] \phi_-(t) &= 0 \end{aligned} \quad (5)$$

where $\Phi(t) = [\phi_+(t) \ \phi_-(t)]^T$. The eigenstates of γ^0 are $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, they are the spinors in the rest frame.

For the mass profile (3), there are two important bases of solutions in which we are interested in. The first one is the ‘in’ basis in which the two independent solutions of the

second order linear differential equations become different single frequency modes in the $t \rightarrow -\infty$ limit. In other words, one solution becomes the negative energy mode and the other solution becomes the positive energy mode. Similarly, there is also an ‘out’ basis of solutions in which one solution becomes the negative energy mode and the other becomes the positive energy mode in the $t \rightarrow \infty$ limit. Accordingly, we will also have different ‘in’ and ‘out’ creation and annihilation operators. Consider the solutions of (5) in the two bases to be

$$\phi_{\pm}(t, k) = \phi_{in, \pm p}(t, k) + \phi_{in, \pm m}(t, k) \quad (6)$$

$$\phi_{\pm}(t, k) = \phi_{out, \pm p}(t, k) + \phi_{out, \pm m}(t, k) \quad (7)$$

where the limits are

$$\begin{aligned} \lim_{t \rightarrow -\infty} \phi_{in, \pm p}(t, k) &= e^{-i\omega_{in}t}, & \lim_{t \rightarrow -\infty} \phi_{in, \pm m}(t, k) &= e^{i\omega_{in}t} \\ \lim_{t \rightarrow \infty} \phi_{out, \pm p}(t, k) &= e^{-i\omega_{out}t}, & \lim_{t \rightarrow \infty} \phi_{out, \pm m}(t, k) &= e^{i\omega_{out}t} \end{aligned}$$

where ‘p’ means *positive energy* and ‘m’ means *negative energy*. The above solutions are the same but written in two different bases for simplicity in the appropriate time limits, they are related by Bogoliubov transformations.

But from (5), we see that the equations of ϕ_+ and ϕ_- are the complex conjugates of each other, so

$$\phi_{in, +p}(t, k) = \phi_{in, -m}^*(t, k), \quad \phi_{in, +m}(t, k) = \phi_{in, -p}^*(t, k) \quad (8)$$

$$\phi_{out, +p}(t, k) = \phi_{out, -m}^*(t, k), \quad \phi_{out, +m}(t, k) = \phi_{out, -p}^*(t, k) \quad (9)$$

The Bogoliubov transformations are

$$\begin{aligned} \phi_{in, +p}(t, k) &= \alpha'_+(k)\phi_{out, +p}(t, k) + \beta'_+(k)\phi_{out, +m}(t, k) \\ &= \alpha'_+(k)\phi_{out, +p}(t, k) + \beta'_+(k)\phi_{out, -p}^*(t, k) \end{aligned} \quad (10)$$

$$\begin{aligned} \phi_{in, -p}(t, k) &= \alpha'_-(k)\phi_{out, -p}(t, k) + \beta'_-(k)\phi_{out, -m}(t, k) \\ &= \alpha'_-(k)\phi_{out, -p}(t, k) + \beta'_-(k)\phi_{out, +p}^*(t, k) \end{aligned} \quad (11)$$

where $\alpha'_{\pm}(k)$ and $\beta'_{\pm}(k)$ are actually functions of $|k|$, since the equations of motion have only k^2 terms.

Now, suppressing the basis labels ‘in’ and ‘out’ since they apply to both bases, we write the u_0 part of $\Psi(x, t)$ as (upto normalization)

$$\tilde{U}(x, t; k) = [\gamma^0 \partial_t + \gamma^1 \partial_x - im(t)] e^{ikx} \phi_{+p}(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (12)$$

And the v_0 part of $\Psi(x, t)$ as

$$\begin{aligned} \tilde{V}(x, t; k) &= [\gamma^0 \partial_t + \gamma^1 \partial_x - im(t)] e^{-ikx} \phi_{-m}(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= [\gamma^0 \partial_t + \gamma^1 \partial_x - im(t)] e^{-ikx} \phi_{+p}^*(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \quad (13)$$

We can define the spinors as (upto normalization)

$$\begin{aligned}\tilde{u}(t, k) &= \frac{1}{e^{ikx}\phi_{+p}(t)}\tilde{U}(x, t; k) \\ \tilde{v}(t, k) &= \frac{1}{e^{-ikx}\phi_{-m}(t)}\tilde{V}(x, t; k)\end{aligned}$$

With proper normalization, the final Dirac fermion mode expansion is

$$\begin{aligned}\Psi(x, t) &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega}} \left[a_k U(x, t; k) + b_k^\dagger V(x, t; k) \right] \\ &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega}} \left[a_k u(t; k) e^{ikx} \phi_{+p}(t) + b_k^\dagger v(t; k) e^{-ikx} \phi_{-m}(t) \right]\end{aligned}\quad (14)$$

2.1 Bogoliubov transformation of oscillators

The initial mass is taken to be $\lim_{t \rightarrow -\infty} m(t) = m$. It is convenient to take the final mass $\lim_{t \rightarrow \infty} m(t)$ to be some m_{out} , because of the spinor convention (in P&S), although we are interested in $m_{out} = 0$.

With time-dependent mass, as mentioned above, the spinors are functionals of $m(t)$, but their normalizations are constants or else they will not solve the Dirac equations. So, we have to differentiate between ‘in’ spinors and ‘out’ spinors. Taking this into account, the mode expansion of $\Psi(x, t)$ starting from ‘in’ basis to ‘out’ basis is

$$\begin{aligned}\Psi(x, t) &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_{in}}} \left[a_{in,k} u_{in}(k, m) \phi_{in,+p}(t, k) e^{ikx} + b_{in,k}^\dagger v_{in}(k, m) \phi_{in,+p}^*(t, k) e^{-ikx} \right] \\ &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_{in}}} \left[\{ \alpha'_+(k) a_{in,k} u_{in}(k, m) \phi_{out,+p}(t, k) + b_{in,-k}^\dagger v_{in}(-k, m) \beta'_+{}^*(k) \phi_{out,-p}(t, k) \} e^{ikx} \right. \\ &\quad \left. + \{ \alpha'_+{}^*(k) b_{in,k}^\dagger v_{in}(k, m) \phi_{out,+p}^*(t, k) + a_{in,-k} u_{in}(-k, m) \beta'_+(k) \phi_{out,-p}^*(t, k) \} e^{-ikx} \right]\end{aligned}\quad (15)$$

where we have used the facts that the k integral is from $-\infty$ to ∞ and α'_\pm , β'_\pm and $\phi_{\pm p}$ are functions of $|k|$. In $t \rightarrow \infty$ limit, $m(t) \rightarrow m_{out}$, so,

$$\begin{aligned}\lim_{t \rightarrow \infty} \Psi(x, t) &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_{out}}} \sqrt{\frac{\omega_{out}}{\omega_{in}}} \left[\{ \alpha'_+(k) a_{in,k} u_{in}(k, m_{out}) + b_{in,-k}^\dagger v_{in}(-k, m_{out}) \beta'_+{}^*(k) \} e^{ikx} \right. \\ &\quad \left. + \{ \alpha'_+{}^*(k) b_{in,k}^\dagger v_{in}(k, m_{out}) + a_{in,-k} u_{in}(-k, m_{out}) \beta'_+(k) \} e^{-ikx} \right]\end{aligned}$$

Comparing with the mode expansion in the ‘out’ solution basis in the same limit $t \rightarrow \infty$,

$$\lim_{t \rightarrow \infty} \Psi(x, t) = \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_{out}}} \left[a_{out,k} u_{out}(k, m_{out}) \phi_{out,+p}(t, k) e^{ikx} + b_{out,k}^\dagger v_{out}(k, m_{out}) \phi_{out,+p}^*(t, k) e^{-ikx} \right]$$

we get the Bogoliubov transformations of the creation and annihilation operators.

$$a_{out,k} = \alpha_+(k) a_{in,k} + b_{in,-k}^\dagger \chi(k, m_{out}) \beta_+^*(k) \quad (16)$$

$$b_{out,k}^\dagger = \alpha_+^*(k) b_{in,k}^\dagger + a_{in,-k} \tilde{\chi}(k, m_{out}) \beta_+(k) \quad (17)$$

where $\alpha_+(k) = \sqrt{\frac{\omega_{out}(\omega_{out}+m_{out})}{\omega_{in}(\omega_{in}+m_{in})}} \alpha'_+$ and $\beta_+(k) = \sqrt{\frac{\omega_{out}(\omega_{out}+m_{out})}{\omega_{in}(\omega_{in}+m_{in})}} \beta'_+(k)$. Using (114)

$$\begin{aligned}\chi(k, m_{out}) &= \frac{1}{2m_{out}} \sqrt{\frac{\omega_{in} + m_{in}}{\omega_{out} + m_{out}}} \bar{u}_{out}(k, m_{out}, \omega_{out}) v_{in}(-k, m_{out}, -\omega_{out}) \\ &= \text{sgn}(k) \quad \text{when } m_{out} \rightarrow 0\end{aligned}\quad (18)$$

where we have to be careful that $v_{in}(k, m_{out})$ is a functional of the accompanying mode, which is $\sim e^{-i\omega_{out}t}$ in the above case. Similarly, with $m_{out} \rightarrow 0$,

$$\tilde{\chi}(k) = -\frac{1}{2m_{out}} \sqrt{\frac{\omega_{in} + m_{in}}{\omega_{out} + m_{out}}} \bar{v}(k, m_{out}, \omega_{out}) u(-k, m_{out}, \omega_{out}) = \text{sgn}(k) \quad (19)$$

taking into account the normalization of $\bar{v}_{out}v_{out} = -2m_{out}$. Inverting (16) and (17), we get

$$a_{in,k} = \alpha_+^*(k) a_{out,k} - \text{sgn}(k) \beta_+^*(k) b_{out,-k}^\dagger \quad (20)$$

$$b_{in,-k}^\dagger = \alpha_+(k) b_{out,-k}^\dagger + \text{sgn}(k) \beta_+(k) a_{out,k} \quad (21)$$

From here on, we will suppress the subscript ‘out’ on creation and annihilation operators, so $a_{out,k} = a_k$, similarly for $b_{out,k}$ and their Hermitian conjugates. Also, since $\chi(k)$ and $\tilde{\chi}(k)$ are simple sign functions, with a slight abuse of the nomenclature, we will call $\alpha_+(k)$ and $\beta_+(k)$ as the Bogoluibov coefficients. Moreover, $\chi(k)^2$ and $\tilde{\chi}(k)^2$ are identically equal to 1. So, the fermionic anti-commutation relations of the ‘in’ and ‘out’ operators constraint the Bogoluibov coefficients as

$$|\alpha_+(k)|^2 + |\beta_+(k)|^2 = 1 \quad (22)$$

2.2 Explicit solutions

In the ‘in’ basis, for our choice of mass profile, the solutions are

$$\begin{aligned}\phi_{in,+p} &= e^{-it(\omega+m)} (e^{-2\rho t} + 1)^{-\frac{im}{2\rho}} {}_2F_1\left(\frac{i(|k| - m - \omega)}{2\rho}, -\frac{i(|k| + m + \omega)}{2\rho}; 1 - \frac{i\omega}{\rho}; -e^{2t\rho}\right) \\ \phi_{in,-m} &= e^{it(\omega-m)} (e^{-2\rho t} + 1)^{-\frac{im}{2\rho}} {}_2F_1\left(\frac{i(-|k| - m + \omega)}{2\rho}, \frac{i(|k| - m + \omega)}{2\rho}; \frac{i\omega}{\rho} + 1; -e^{2t\rho}\right)\end{aligned}$$

where $\omega = \sqrt{k^2 + m^2}$. While in the ‘out’ basis, the solutions are

$$\begin{aligned}\phi_{out,+p} &= e^{-i|k|t} (e^{-2\rho t} + 1)^{-\frac{im}{2\rho}} {}_2F_1\left(\frac{i|k| - im + i\omega}{2\rho}, \frac{i|k| - im - i\omega}{2\rho}; 1 + \frac{i|k|}{\rho}; -e^{-2\rho t}\right) \\ \phi_{out,-m} &= e^{i|k|t} (e^{-2\rho t} + 1)^{-\frac{im}{2\rho}} {}_2F_1\left(\frac{-i|k| - im + i\omega}{2\rho}, \frac{-i|k| - im - i\omega}{2\rho}; 1 - \frac{i|k|}{\rho}; -e^{-2\rho t}\right)\end{aligned}$$

Using the properties of confluent hypergeometric functions ${}_2F_1$ given in [25], the Bogoli-

ubov coefficients of the frequency modes as defined in (10) are

$$\alpha'_+ = \frac{\Gamma\left(-\frac{i|k|}{\rho}\right) \Gamma\left(1 - \frac{i\omega}{\rho}\right)}{\Gamma\left(-\frac{i(|k|+m+\omega)}{2\rho}\right) \Gamma\left(1 + \frac{-i|k|+im-i\omega}{2\rho}\right)} \quad (23)$$

$$\beta'_+ = \frac{\Gamma\left(\frac{i|k|}{\rho}\right) \Gamma\left(1 - \frac{i\omega}{\rho}\right)}{\Gamma\left(1 - \frac{i(-|k|-m+\omega)}{2\rho}\right) \Gamma\left(-\frac{i(-|k|+m+\omega)}{2\rho}\right)} \quad (24)$$

In the sudden limit($\rho \rightarrow \infty$), the Bogoliubov coefficients of the frequency modes are

$$\alpha'_+(k) = \frac{|k| + m_{in} + \sqrt{k^2 + m^2}}{2|k|} \quad (25)$$

$$\beta'_+(k) = \frac{|k| - m_{in} - \sqrt{k^2 + m^2}}{2|k|} \quad (26)$$

As mentioned above, for a quench starting from the ground state of the massive theory, the naive sudden limit gives a non-normalizable state in the massless theory [1]. The problem arises only for a quench starting from the ground state. In case the quench is starting from squeezed states of our interest, the naive sudden limit given above works well. As defined in (16) and (17), the Bogoliubov coefficients of the oscillator modes differ from $\alpha'_+(k)$ and $\beta'_+(k)$ by an overall factor.

$$\alpha_+ = \sqrt{1 - \frac{m}{\sqrt{k^2 + m^2}}} \frac{\Gamma\left(-\frac{i|k|}{\rho}\right) \Gamma\left(1 - \frac{i\omega}{\rho}\right)}{\Gamma\left(-\frac{i(|k|+m+\omega)}{2\rho}\right) \Gamma\left(1 + \frac{-i|k|+im-i\omega}{2\rho}\right)} \quad (27)$$

$$\beta_+ = \sqrt{1 - \frac{m}{\sqrt{k^2 + m^2}}} \frac{\Gamma\left(\frac{i|k|}{\rho}\right) \Gamma\left(1 - \frac{i\omega}{\rho}\right)}{\Gamma\left(1 - \frac{i(-|k|-m+\omega)}{2\rho}\right) \Gamma\left(-\frac{i(-|k|+m+\omega)}{2\rho}\right)} \quad (28)$$

In the sudden limit, they are

$$\alpha_+(k) = \sqrt{1 - \frac{m}{\sqrt{k^2 + m^2}}} \frac{|k| + m + \sqrt{k^2 + m^2}}{2|k|} \quad (29)$$

$$\beta_+(k) = \sqrt{1 - \frac{m}{\sqrt{k^2 + m^2}}} \frac{|k| - m - \sqrt{k^2 + m^2}}{2|k|} \quad (30)$$

For completeness, the expressions of α'_- and β'_- in (11) for our particular quench protocol are

$$\alpha'_- = \frac{\Gamma\left(-\frac{i|k|}{\rho}\right) \Gamma\left(1 - \frac{i\omega}{\rho}\right)}{\Gamma\left(-\frac{i(|k|-m+\omega)}{2\rho}\right) \Gamma\left(1 - \frac{i(|k|+m+\omega)}{2\rho}\right)} \quad (31)$$

$$\beta'_- = \frac{\Gamma\left(\frac{i|k|}{\rho}\right) \Gamma\left(1 - \frac{i\omega}{\rho}\right)}{\Gamma\left(\frac{i(|k|+m-\omega)}{2\rho}\right) \Gamma\left(1 - \frac{i(-|k|+m+\omega)}{2\rho}\right)} \quad (32)$$

3 Quenched states

3.1 From ground state

Starting from the ground state of the massive theory $|\Psi\rangle = |0, in\rangle$, using Eq (20), the state in terms of ‘out’ operators is given by

$$\begin{aligned} a_{in,k}|\Psi\rangle = 0 &\Rightarrow \left[\alpha_+^*(k)a_k - \text{sgn}(k)\beta_+^*(k)b_{-k}^\dagger \right] |\Psi\rangle = 0 \\ &\Rightarrow |\Psi\rangle = e^{\sum_k \text{sgn}(k)\gamma(k)a_k^\dagger b_{-k}^\dagger} |0\rangle \end{aligned} \quad (33)$$

$$\text{where } \gamma(k) = \frac{\alpha_+^*(k)}{\beta_+^*(k)} \quad (34)$$

where we have taken $|0\rangle$ to be the ground state of ‘out’ oscillators. Using the Baker-Campbell-Hausdorff(BCH) formula derived in appendix (E), the above state can be written in gCC form. For the particular mass profile (3), $\alpha_+(k)$ and $\beta_+(k)$ are given in (27) and (28). The gCC form which was first obtained in MPS is

$$|\Psi\rangle = e^{-\kappa_2 H - \kappa_4 W_4 - \kappa_6 W_6 - \dots} |D\rangle \quad (35)$$

where

$$\begin{aligned} \kappa_2 &= \frac{1}{2m} + \frac{\pi^2 m}{12\rho^2} + \frac{1}{m} \mathcal{O}\left(\frac{m}{\rho}\right)^3, \quad \kappa_4 = -\frac{1}{12m^3} + \frac{\pi^2}{24m\rho^2} + \frac{1}{m^3} \mathcal{O}\left(\frac{m}{\rho}\right)^3, \\ \kappa_6 &= \frac{3}{80m^5} - \frac{\pi^2}{96m^3\rho^2} + \frac{1}{m^5} \mathcal{O}\left(\frac{m}{\rho}\right)^3, \dots \end{aligned} \quad (36)$$

and $|D\rangle$ is the Dirichelet state and the explicit expression is in Appendix C. It should be noted that since the mass does not go to zero at any finite time, the above state should be only valid in sufficiently long time limit and the correction due to the non-vanishing mass is $\mathcal{O}(e^{-\rho t})$.

3.2 From squeezed states: CC state and gCC states

We could start with specially prepared squeezed states so that after the quench, the states become CC states or gCC states. Here, we will consider only the simple case of sudden quench ($\rho \rightarrow \infty$). For our aim of creating a CC state or a gCC state, finite ‘ ρ ’ quenches are an unnecessary complication.

We start with a squeezed state of ‘in’ modes

$$|S\rangle = \exp\left(\sum_{k=-\infty}^{\infty} f(k)a_{in,k}^\dagger b_{in,-k}^\dagger\right) |0, in\rangle \quad (37)$$

where unlike $\gamma(k)$, $f(k)$ need not be an even function of k , but $|f(k)|^2$ is an even function of k .

It is easier to work with $|S\rangle$ as an operator relation. $|S\rangle$ can also be defined as

$$\tilde{a}_k|S\rangle = \tilde{b}_k|S\rangle = 0 \text{ and } \left\{ \tilde{a}_k, \tilde{a}_{k'}^\dagger \right\} = \left\{ \tilde{b}_{-k}, \tilde{b}_{-k'}^\dagger \right\} = \delta(k - k') \quad (38)$$

where the new operators in terms of the out modes using (20) and (21) are

$$\begin{aligned} \tilde{a}_k &= \frac{1}{\sqrt{(1 + |f(k)|^2)}} a_{in,k} - \frac{f(k)}{\sqrt{(1 + |f(k)|^2)}} b_{in,-k}^\dagger \\ &= A^*(k) a_{out,k} - \text{sgn}(k) B^*(k) b_{out,-k} \\ \tilde{b}_{-k} &= \frac{1}{\sqrt{(1 + |f(k)|^2)}} b_{in,-k} + \frac{f(k)}{\sqrt{(1 + |f(k)|^2)}} a_{in,k}^\dagger \\ &= A^*(k) b_{out,-k} + \text{sgn}(k) B^*(k) a_{out,k}^\dagger \end{aligned} \quad (39)$$

where $A(k)$ and $B(k)$ are the Bogoliubov coefficients for the transformation from ‘*tilde*’ operators to ‘out’ operators and are given by

$$A(k) = \frac{\alpha_+(k) - \text{sgn}(k) \beta_+^*(k) f^*(k)}{\sqrt{(1 + |f(k)|^2)}}, \quad B(k) = \frac{\beta_+(k) + \text{sgn}(k) \alpha_+^*(k) f^*(k)}{\sqrt{(1 + |f(k)|^2)}} \quad (40)$$

$$|A(k)|^2 + |B(k)|^2 = 1 \quad (41)$$

Now using the BCH formula (133) from appendix (E),

$$\begin{aligned} |S\rangle &= \exp \left\{ - \sum_k \tilde{\kappa}(k) \left(a_{out,k}^\dagger a_{out,k} + b_{out,k}^\dagger b_{out,k} \right) \right\} |D\rangle \\ \text{where } \tilde{\gamma}(k) &= \frac{B^*(k)}{A^*(k)}, \quad \text{and } \tilde{\kappa}(k) = -\frac{1}{2} \log(\tilde{\gamma}(k)) \end{aligned} \quad (42)$$

For a CC state, i.e., so that $|S\rangle$ in eqn (42) is $e^{-\kappa_2 H} |D\rangle$, $f(k)$ should be tuned as

$$f(k) = \frac{(\sqrt{k^2 + m^2} + m) \cosh(\kappa_2 k) - k \sinh(\kappa_2 k)}{(\sqrt{k^2 + m^2} + m) \sinh(\kappa_2 k) + k \cosh(\kappa_2 k)} \quad (43)$$

Starting with

$$f(k) = \frac{k - k e^{2|k|(\kappa_2 + \kappa_4 k^2)} + \text{sgn}(k) (\sqrt{k^2 + m^2} + m) (e^{2|k|(\kappa_2 + \kappa_4 k^2)} + 1)}{|k| (e^{2|k|(\kappa_2 + \kappa_4 k^2)} + 1) + (\sqrt{k^2 + m^2} + m) (e^{2|k|(\kappa_2 + \kappa_4 k^2)} - 1)} \quad (44)$$

we get a gCC state of the form $e^{-\kappa_2 H - \kappa_4 W_4} |D\rangle$, where as mentioned earlier, W_4 is the conserved charge of the W_4 current of free Dirac fermions³. Note that $f(k)$ are odd functions of

³For the action (1), $H = \sum_k |k| (a_k^\dagger a_k + b_k^\dagger b_k)$ or $H = \int \frac{dk}{2\pi} |k| (a_k^\dagger a_k + b_k^\dagger b_k)$. W_4 has been normalized so that $W_4 = \sum_k |k|^3 (a_k^\dagger a_k + b_k^\dagger b_k)$ or $W_4 = \int \frac{dk}{(2\pi)^3} |k|^3 (a_k^\dagger a_k + b_k^\dagger b_k)$ in the continuum limit.

k . For future reference, we can invert Eq (39) and we write down the ‘in’ and ‘out’ operators in terms of the ‘*tilde*’ operators.

$$a_{in,k} = \frac{1}{\sqrt{(1+|f(k)|^2)}} \tilde{a}_k + \frac{f(k)}{\sqrt{(1+|f(k)|^2)}} \tilde{b}_{-k}^\dagger \quad (45)$$

$$b_{in,-k}^\dagger = \frac{1}{\sqrt{(1+|f(k)|^2)}} \tilde{b}_{-k}^\dagger - \frac{f^*(k)}{\sqrt{(1+|f(k)|^2)}} \tilde{a}_k \quad (46)$$

$$a_{out,k} = A(k) \tilde{a}_k + \text{sgn}(k) B^*(k) \tilde{b}_{-k}^\dagger \quad (47)$$

$$b_{out,-k}^\dagger = A^*(k) \tilde{b}_{-k}^\dagger - \text{sgn}(k) B(k) \tilde{a}_k \quad (48)$$

4 Energy density

In the post-quench theory, the occupation number is given by

$$\hat{N}_k = a_k^\dagger a_k + b_k^\dagger b_k \quad (49)$$

using the Bogoliubov transformations (47) and (48) and definition of $|\tilde{0}\rangle$ in (38), the expectation value of the occupation number is given by

$$\begin{aligned} N_k &= \lim_{t \rightarrow \infty} \langle \tilde{0} | a_k^\dagger a_k + b_k^\dagger b_k | \tilde{0} \rangle \\ &= B^*(k) B(k) \langle \tilde{0} | \tilde{b}_{-k}^\dagger \tilde{b}_{-k} | \tilde{0} \rangle + B^*(-k) B(-k) \langle \tilde{0} | \tilde{a}_{-k}^\dagger \tilde{a}_{-k} | \tilde{0} \rangle + \dots \\ &= B^*(k) B(k) + B^*(-k) B(-k) \end{aligned} \quad (50)$$

The expression of $B(k)$ is given in (40). For ground state, we have to use $f(k) = 0$ in the expression of $B(k)$. So, energy density of the post-quench state is given by

$$E = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k| [B^*(k) B(k) + B^*(-k) B(-k)] \quad (51)$$

Ground state quench

For ground state quench, the occupation number is given by

$$N_k = \lim_{t \rightarrow \infty} \langle 0, in | \hat{N}_k | 0, in \rangle = |\beta_+(k)|^2 + |\beta_+(-k)|^2$$

Since $\alpha_+(k)$ and $\beta_+(k)$ are even functions of k . Using (22), (34) and (134), we have

$$|\beta_+(k)|^2 = \frac{|\gamma(k)|^2}{1 + |\gamma(k)|^2}, \quad |\gamma(k)|^2 = e^{-4\kappa(k)} \quad (52)$$

Hence, the occupation number in the ground state in the asymptotically long time limit is given by

$$N_k = \frac{2}{e^{4\kappa(k)} + 1} \quad (53)$$

This is the occupation number in a GGE defined as

$$\text{Tr} e^{-\sum_k 4\kappa(k)\hat{N}_k} = \text{Tr} e^{-4\kappa_2 H - 4\kappa_4 W_4 - \kappa_6 W_6 - \dots} \quad (54)$$

where the κ 's are given in (36). Using the expressions of $\beta_+(k)$ from (28), the explicit expression of the occupation number is

$$N_k = \text{csch}\left(\frac{\pi k}{\rho}\right) \left(\cosh\left(\frac{\pi m}{\rho}\right) - \cosh\left(\frac{\pi(k - \sqrt{k^2 + m^2})}{\rho}\right) \right) \text{csch}\left(\frac{\pi\sqrt{k^2 + m^2}}{\rho}\right)$$

$$\xrightarrow{\rho \rightarrow \infty} 1 - \frac{k}{\sqrt{k^2 + m^2}}$$

It is interesting that in $m \rightarrow \infty$ limit, $N_k \rightarrow 1$, not 2. This is because $\lim_{\rho \rightarrow \infty} |\alpha_+(k)|^2 = 1/2$ and we have the constraint $|\alpha_+(k)|^2 + |\beta_+(k)|^2 = 1$.

For arbitrary ρ , the energy density cannot be calculated in closed form. In the sudden limit $\rho \rightarrow \infty$, the energy density diverges as $\log(\Lambda)$ where Λ is the UV cutoff. Hence, all other W charges also diverge in the sudden limit. Hence, naively taking $\rho \rightarrow \infty$ produce a non-renormalizable state. So, the sudden limit has to be taken as in MPS where $m/\Lambda \rightarrow 0$ while $m/\rho \rightarrow \epsilon^+$. Simply put, the quench rate parameter ρ should be much small than the UV cut-off.

Squeezed state quench: CC and gCC states

For CC state given by (43), the expectation value of occupation number is given by

$$N_k = \frac{2}{1 + e^{4\kappa_2|k|}} \quad (55)$$

This is the occupation number of fermions in a thermal ensemble of temperature $1/\beta = 1/4\kappa_2$. The energy density is

$$E = \int_{-\infty}^{\infty} \frac{dk}{2\pi} N_k = \frac{\pi}{96\kappa_2^2} \quad (56)$$

Similarly, for gCC state given by (44), the expectation value of occupation number is given by

$$N_k = \langle gCC | \hat{N}_k | gCC \rangle = \frac{2}{1 + e^{4\kappa_2|k| + 4\kappa_4|k|^3}} \quad (57)$$

This is same as the occupation number of fermions in a generalised Gibbs ensemble of temperature $1/\beta = 4\kappa_2$ and chemical potential $\mu_4 = 4\kappa_4$ of W_4 charge. The energy density cannot be calculated in closed form.

5 Correlation functions

Since our theory is a free theory, all the observables can be explicitly calculated. In the following subsections we calculate $\langle \psi^\dagger(r, t) \psi(0, t) \rangle$ correlation functions for the three different states obtained above. The quench process cannot differentiate between holomorphic dof(‘left-movers’) and anti-holomorphic dof(‘right-movers’), so $\langle \bar{\psi}^\dagger(0, t) \bar{\psi}(r, t) \rangle$ is equal to $\langle \psi^\dagger(r, t) \psi(0, t) \rangle$ and they are time independent quantities.⁴ We also calculated $\langle \bar{\psi}^\dagger(r, t) \psi(0, t) \rangle$ which has non-trivial time-dependence. Also as expected, $-\langle \psi^\dagger(0, t) \bar{\psi}(r, t) \rangle$ is the complex conjugate of $\langle \bar{\psi}^\dagger(r, t) \psi(0, t) \rangle$. Since, we are calculating equal-time correlation functions, so for example for $\langle \psi^\dagger(r, t) \psi(0, t) \rangle$, we would rather be calculating $\frac{1}{2} \langle \psi^\dagger(r, t) \psi(0, t) - \psi(0, t) \psi^\dagger(r, t) \rangle$.

Using the Bogoluibov transformations (47) and (48) in the chiral mode expansions (116) and (117) we get

$$\psi(w) = \int_0^\infty \frac{dk}{2\pi} \left[A(k) \tilde{a}_k e^{-ikw} + \text{sgn}(k) B^*(k) \tilde{b}_{-k}^\dagger e^{-ikw} + A^*(-k) \tilde{b}_k^\dagger e^{ikw} + \text{sgn}(k) B(-k) \tilde{a}_{-k} e^{ikw} \right] \quad (58)$$

$$\bar{\psi}(\bar{w}) = \int_0^\infty \frac{dk}{2\pi} \left[A(-k) \tilde{a}_{-k} e^{-ik\bar{w}} - \text{sgn}(k) B^*(-k) \tilde{b}_k^\dagger e^{-ik\bar{w}} - A^*(k) \tilde{b}_{-k}^\dagger e^{ik\bar{w}} + \text{sgn}(k) B(k) \tilde{a}_k e^{ik\bar{w}} \right] \quad (59)$$

where $w = t - x$ and $\bar{w} = t + x$. For the ground state quench, $f(k) = 0$, $\tilde{a}_k = a_{in,k}$, $\tilde{b} = b_{in,k}$ and $|\tilde{0}\rangle = |0, in\rangle$.

For a general $f(k)$ corresponding to some $|\tilde{0}\rangle$, the correlation functions are

$$\langle \tilde{0} | \psi^\dagger(0, t) \psi(r, t) | \tilde{0} \rangle = \frac{1}{2} \int_0^\infty \frac{dk}{2\pi} \left[(2|B(k)|^2 - 1) e^{ikr} - (2|B(-k)|^2 - 1) e^{-ikr} \right] \quad (60)$$

$$\langle \tilde{0} | \bar{\psi}^\dagger(0, t) \psi(r, t) | \tilde{0} \rangle = - \int_0^\infty \frac{dk}{2\pi} \left[\text{sgn}(k) A^*(-k) B(-k) e^{ik(2t-r)} + \text{sgn}(k) A(k) B^*(k) e^{-ik(2t-r)} \right] \quad (61)$$

where we have used (41) to write $A(k)$ in terms of $B(k)$ in the first equation.

⁴A simple reason why these quantities are time independent is the fact that they are holomorphic-holomorphic and antiholomorphic-antiholomorphic quantities and they cannot ‘see’ the presence of the boundary state $|D\rangle$. They are already thermalized/equilibrated.

Ground state quench:

Taking careful limit, for ground state quench, we have

$$\begin{aligned}
\langle 0, in | \psi^\dagger(0, t) \psi(r, t) | 0, in \rangle &= -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{|k|}{\sqrt{k^2 + m^2}} \\
&= \frac{1}{4} m [\mathbf{L}_{-1}(mr) - I_1(mr)] \\
&\xrightarrow{m \rightarrow \infty} \frac{1}{2\pi m r^2} + \frac{3}{2\pi m^3 r^4} + O\left(\frac{1}{m^4}\right)
\end{aligned} \tag{62}$$

$$\begin{aligned}
\langle \tilde{0} | \bar{\psi}^\dagger(0, t) \psi(r, t) | \tilde{0} \rangle &= \int_0^\infty \frac{dk}{2\pi} \frac{i \operatorname{sgn}(k) m \sin(k(2t - r))}{\sqrt{k^2 + m^2}} \\
&= -\frac{im}{4} [\operatorname{sgn}(r - 2t) I_0(m(r - 2t)) - \mathbf{L}_0(m(r - 2t))] \\
&\xrightarrow[t > r/2]{m \rightarrow \infty} \frac{i}{2\pi(2t - r)} + \frac{i}{2\pi m^2(2t - r)^3} + O\left(\frac{1}{m^4}\right)
\end{aligned} \tag{63}$$

where $I_\nu(x)$ is Modified Bessel Function of the First Kind and $\mathbf{L}_\nu(x)$ is Modified Struve Function.

Quenched squeezed state - CC state:

For CC state, all the calculations are done in $|S_{CC}\rangle$ defined as the state (37) with the expression of $f(k)$ given in (43).

$$\langle CC | \psi^\dagger(0, t) \psi(r, t) | CC \rangle = -i \int_0^\infty \frac{dk}{2\pi} \tanh(2\kappa_2 |k|) \sin(kr) \tag{64}$$

$$= -i \int_0^\infty \frac{dk}{2\pi} \sin(kr) \left[\frac{1}{e^{4\kappa_2 |k|} + 1} - \frac{1}{2} \right] \tag{65}$$

$$= -\frac{i \operatorname{csch}\left(\frac{\pi r}{4\kappa_2}\right)}{8\kappa_2} \tag{66}$$

$$\langle CC | \bar{\psi}^\dagger(0, t) \psi(r, t) | CC \rangle = -i \int_0^\infty \frac{dk}{2\pi} \operatorname{sech}(2k\kappa_2) \cos(k(2t - r)) \tag{67}$$

$$= -\frac{i \operatorname{sech}\left(\frac{\pi(2t-r)}{4\kappa_2}\right)}{8\kappa_2} \tag{68}$$

These are exactly what have been calculated using BCFT techniques [26]. It is evident from (65) that $\psi^\dagger \psi$ expectation value is already the thermal expectation value at temperature $T = 1/\beta = 1/(4\kappa_2)$, i.e., it is already thermalized.

Quenched squeezed state - gCC state with W_4 :

Similarly, for gCC state, all the calculations are done in $|S_{fCC}\rangle$ defined as the state (37) with the expression of $f(k)$ given in (44).

$$\langle \psi^\dagger(0, t) \psi(r, t) \rangle_{gCC} = -i \int_0^\infty \frac{dk}{2\pi} \tanh(2\kappa_2|k| + 2\kappa_4|k|^3) \sin(kr) \quad (69)$$

$$= -i \int_0^\infty \frac{dk}{2\pi} \sin(kr) \left[\frac{1}{e^{4\kappa_2|k| + 4\kappa_4|k|^3} + 1} - \frac{1}{2} \right] \quad (70)$$

$$\langle \bar{\psi}^\dagger(r, t) \psi(0, t) \rangle_{gCC} = -i \int_0^\infty \frac{dk}{2\pi} \text{sech}(2\kappa_2k + 2\kappa_4k^3) \cos(k(2t - r)) \quad (71)$$

Again, it is evident from (70) that $\psi^\dagger\psi$ expectation value is already thermalized into the expectation value in a GGE with $T = 1/\beta = 1/4\kappa_2$ and $\mu = 4\kappa_4$. A possible way of evaluating these integrals (which yield no closed form answer) is via the residue theorem. The integrands in both cases, have poles at the solutions of $2\kappa_2k + 2\kappa_4k^3 = \frac{2n+1}{2}i\pi$, where $n \in \mathbb{Z}$. These poles and their residues have been treated in detail in [1]. The sum of residues is still an infinite sum which cannot be performed. In the perturbative regime ($\kappa_4/\kappa_2^3 \ll 1$), we see that our correlators match the general form presented in [8] with $h = 1/2$ as expected.

As expected from MPS, here in the fermionic theory also we see the UV/IR mixing. For the ground state quench, all the charges affect the long distance and long time limit of the correlators. This is explicit seen in the case of gCC state with W_4 charge only. The long time and large distance limit of the correlators are very much dependent upon k_4 , although a naive Wilsonian RG argument would show that k_4 is an irrelevant coupling.

6 Exact Growth of Entanglement in CC state

We will consider only a finite single interval or subsystem A, with its endpoints at (w_1, \bar{w}_1) and (w_2, \bar{w}_2) in light-cone coordinates, or $(0, t)$ and (r, t) in space and time coordinates. Using the replica trick ([27], [28]), the n^{th} Rényi entropy $S_n(A)$ of the interval is given by the logarithm of the expectation value of twist and antitwist operators inserted at the end-points.

$$S_n(A) = \frac{1}{1-n} \log \langle \Psi(t) | \mathcal{T}_n(w_1, \bar{w}_1) \tilde{\mathcal{T}}_n(w_2, \bar{w}_2) | \Psi(t) \rangle \quad (72)$$

The entanglement entropy (EE) S_A is given by $\lim_{n \rightarrow 1} S_n(A)$. We can diagonalize the twist operators and write them as products of twist fields. Hence,

$$\mathcal{T}_n(w, \bar{w}) = \prod_{k=-(n-1)/2}^{k=(n-1)/2} \mathcal{T}_{k,n}(w, \bar{w}), \quad \tilde{\mathcal{T}}_n(w, \bar{w}) = \prod_{k=-(n-1)/2}^{k=(n-1)/2} \tilde{\mathcal{T}}_{k,n}(w, \bar{w}) \quad (73)$$

In CC state, in Heisenberg picture, the quantity of our interest is

$$Z_k = \langle D_f | e^{-\kappa_2 H_f} \mathcal{T}_{k,n}(0, t) \tilde{\mathcal{T}}_{k,n}(r, t) e^{-\kappa_2 H_f} | D_f \rangle \quad (74)$$

The subscript ‘f’ means we are working in the fermionic theory and the subscript ‘b’ would mean we are working in the bosonic theory. To find the exact expression of the entanglement entropy of a spatial region in our free fermionic CFT, we will use the method using bosonization described in [29]. Moreover, as shown in Appendix(D), Dirichlet state $|D_f\rangle$ in fermionic theory corresponds to a Dirichlet state in the bosonic theory $|D_b\rangle$ and H_f corresponds to H_b . So, we get

$$Z_k = \langle D_b | e^{-\kappa_2 H_b} e^{i\sqrt{4\pi}\frac{k}{n}(\phi(0,t)-\phi(r,t))} e^{-\kappa_2 H_b} | D_b \rangle \quad (75)$$

This is a free scalar theory in a strip geometry with Dirichlet boundary conditions and operator insertions at $(0, t)$ and (r, t) . It can be calculated explicitly

$$\log [Z_k] = -4\pi \frac{2k^2}{n^2} (\langle \phi(0, t) \phi(0, t) \rangle - \langle \phi(0, t) \phi(r, t) \rangle) \quad (76)$$

The n^{th} Rényi entropy of interval A is given by

$$\begin{aligned} S_n(A) &= -4\pi \frac{1}{1-n} \sum_{k=-(n-1)/2}^{k=(n-1)/2} \frac{2k^2}{n^2} (\langle \phi(0, t) \phi(0, t) \rangle - \langle \phi(0, t) \phi(r, t) \rangle) \\ &= 4\pi \frac{n+1}{6n} (\langle \phi(0, t) \phi(0, t) \rangle - \langle \phi(0, t) \phi(r, t) \rangle) \end{aligned} \quad (77)$$

Taking $n \rightarrow 1$ limit, we get the entanglement entropy,

$$S_A = 4\pi \frac{1}{3} (\langle \phi(0, t) \phi(0, t) \rangle - \langle \phi(0, t) \phi(r, t) \rangle) \quad (78)$$

Remark on winding number: While the free boson considered in MPS [1] is the uncompactified free boson, the boson in (75) is a compactified free boson. So, Hamiltonian of the compactified boson has zero mode terms but the winding number is not important for our analysis. In the large system size limit ($L \rightarrow \infty$), the zero modes vanished. Even if we are taking the limiting case of a finite size system, the zero momentum modes do not play any role in our calculation. Using the mode expansion of the boson $\phi(w, \bar{w}) = \varphi(w) + \bar{\varphi}(\bar{w})$ in [30],

$$\varphi(w) = Q + \frac{P}{2L} w + \sum_{n>0} \frac{1}{\sqrt{4\pi n}} (d_n e^{-inw} + d_n^\dagger e^{inw}) \quad (79)$$

$$\bar{\varphi}(\bar{w}) = \bar{Q} + \frac{\bar{P}}{2L} \bar{w} + \sum_{n>0} \frac{1}{\sqrt{4\pi n}} (d_{-n} e^{-in\bar{w}} + d_{-n}^\dagger e^{in\bar{w}}) \quad (80)$$

First, Q and \bar{Q} are cancelled identically in (75). Moreover, by bosonization formulae [30, 31],

$$P = \sqrt{4\pi} N_f \quad \bar{P} = \sqrt{4\pi} \bar{N}_f \quad (81)$$

$$N_f = J_0 = - \sum_{k=0}^{\infty} [a_k^\dagger a_k - b_k^\dagger b_k] \quad \bar{N}_f = \bar{J}_0 = - \sum_{k=0}^{\infty} [a_{-k}^\dagger a_{-k} - b_{-k}^\dagger b_{-k}] \quad (82)$$

But for our particular CC state, from (131), $N_f |CC_f\rangle = 0$ and $\bar{N}_f |CC_f\rangle = 0$. Now, P and \bar{P} commute with all the other bosonic creation and annihilation operators of non-zero momentum,

hence they don't play any role in the calculation of (75). If we still keep the system size finite, the winding number would be important to interpret the stationary limit as a thermal ensemble. But we must take the $L \rightarrow \infty$ limit, if we want to examine the stationary limit. In other words, L is the largest length scale in our theory and time $t \ll L$.

The bosonic propagator in CC state has been calculated in [1]. It is given by

$$\langle CC | \phi(0, t) \phi(r, t) | CC \rangle = -\frac{1}{8\pi} \log \left(\frac{2 \sinh^2 \left(\frac{\pi r}{4\kappa_2} \right)}{\cosh \left(\frac{\pi r}{2\kappa_2} \right) + \cosh \left(\frac{\pi t}{\kappa_2} \right)} \right) \quad (83)$$

$r \rightarrow 0$ gives the UV divergence of scalar field theory in 2D spacetime. The n^{th} Rényi entropy and entanglement entropy of interval A in CC state is given by

$$S_n(A) = \frac{n+1}{12n} \left[\log \left(\frac{\sinh^2 \left(\frac{\pi r}{4\kappa_2} \right) \left(1 + \cosh \left(\frac{\pi t}{\kappa_2} \right) \right)}{\cosh \left(\frac{\pi r}{2\kappa_2} \right) + \cosh \left(\frac{\pi t}{\kappa_2} \right)} \right) - \lim_{\epsilon \rightarrow 0+} 2 \log(\epsilon) - \log \left(\frac{\pi^2}{16\kappa_2^2} \right) \right] \quad (84)$$

$$S_A = \frac{1}{3} \left[\frac{1}{2} \log \left(\frac{\sinh^2 \left(\frac{\pi r}{4\kappa_2} \right) \left(1 + \cosh \left(\frac{\pi t}{\kappa_2} \right) \right)}{\cosh \left(\frac{\pi r}{2\kappa_2} \right) + \cosh \left(\frac{\pi t}{\kappa_2} \right)} \right) - \lim_{\epsilon \rightarrow 0+} \log(\epsilon) - \frac{1}{2} \log \left(\frac{\pi^2}{16\kappa_2^2} \right) \right] \quad (85)$$

Taking the stationary limit $t \rightarrow \infty$ gives the entanglement entropy of A in a thermal ensemble at temperature $T = 1/\beta = 1/(4\kappa_2)$.

$$S_A = \frac{1}{3} \left[\log \left(\sinh \left(\frac{\pi r}{4\kappa_2} \right) \right) - \lim_{\epsilon \rightarrow 0+} \log(\epsilon) - \frac{1}{2} \log \left(\frac{\pi^2}{16\kappa_2^2} \right) \right] \quad (86)$$

This exactly matches the thermal value which has been calculated using CFT techniques [28]. It is fixed only by the temperature, the central charge of the CFT ($c = 1$ for dirac fermion), and 'r' the length of the interval. Taking the high temperature limit $\kappa_2 \rightarrow 0$, we get the extensive thermal entropy formula $S_{\text{therm}} = \frac{1}{3} \frac{\pi r}{\beta}$.

Besides the thermalization, the most interesting aspect of figure (1) is that the entanglement entropy grows monotonically. The first derivative of S_A w.r.t. time is

$$\left\langle \frac{\partial S_A}{\partial t} \right\rangle_{CC} = \frac{\pi \sinh^2 \left(\frac{\pi r}{4\kappa_2} \right) \tanh \left(\frac{\pi t}{2\kappa_2} \right)}{3\kappa_2 \left[\cosh \left(\frac{\pi r}{2\kappa_2} \right) + \cosh \left(\frac{\pi t}{\kappa_2} \right) \right]} \quad (87)$$

$$= \frac{\pi}{12\kappa_2} \left[2 \tanh \left(\frac{\pi t}{2\kappa_2} \right) - \tanh \left(\frac{\pi(r+2t)}{4\kappa_2} \right) + \tanh \left(\frac{\pi(r-2t)}{4\kappa_2} \right) \right] \quad (88)$$

From the first expression, as a function of time $t > 0$, it is clear that there are no finite zero. Hence, the EE growth of CC state is always monotonically increasing. Also note that in the high effective temperature limit $\kappa_2 \rightarrow 0$, the approach to thermal value is sharper. In the limiting case, from the second expression, it is clear that the thermalization time is

$$t = \frac{r}{2} \quad (89)$$

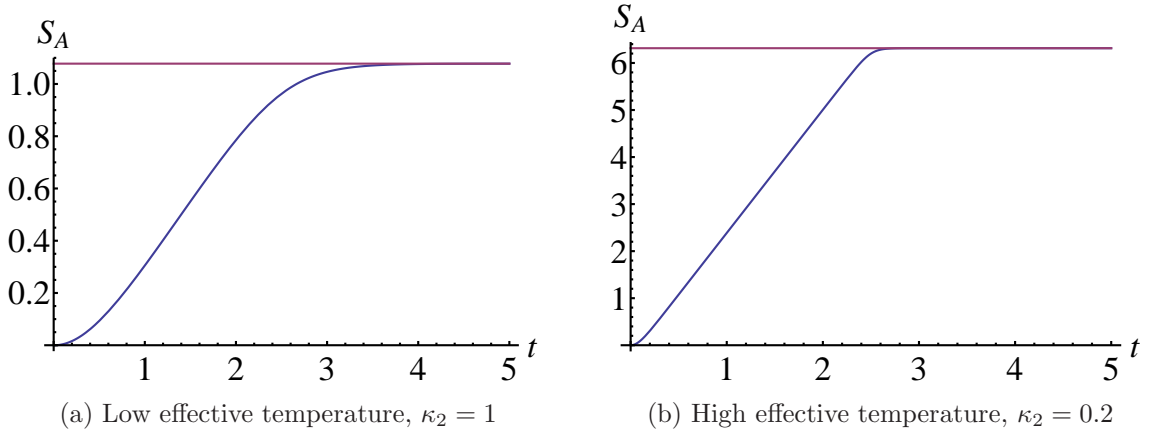


Figure 1: Entanglement entropy growth of an interval($r=5$) in CC state.

which has also been calculated using BCFT techniques in [6].

It would be interesting to check the monotonicity of EE growth in gCC states. Unfortunately, even for the free fermions with explicit twist operators, the entanglement entropy in gCC state with W_4 charge cannot be explicitly calculated. The bilinear fermionic $\mathcal{W}_4(w)$ current when bosonized gives ϕ^4 terms[10], so the bosonized theory is an interacting theory.

7 Non-Monotonic EE Growth and Dynamical Phase Transition

Although we could not calculate EE in gCC state with W_4 charge of the fermionic bilinear \mathcal{W}_4 current, we can still calculate entanglement entropy explicitly with the fermionic charge corresponding to the bosonic charge $W_4(w) = \sum_k |k|^3 d_k^\dagger d_k$, where d_k^\dagger and d_k are the bosonic annihilation and creation operators. As mentioned above, the zero modes do not play any role. Reformionization of the bosonic bilinear \mathcal{W}_4 is done in Appendix F.⁵ So, the fermionic state that we are considering is

$$|\Psi\rangle = e^{-\kappa_2 H_f - \kappa_4 \tilde{W}_4} |D_f\rangle \quad (90)$$

where the expression for \tilde{W}_4 is given in (137).

Again, the Rényi and entanglement entropies are given by the expression (77) and (78). The scalar propagator with the bosonic W_4 charge has also been calculated in MPS.

$$\langle \phi(0, t) \phi(r, t) \rangle = \int_{-\infty}^{\infty} \frac{dk}{8\pi} \frac{e^{ikr}}{k} [\coth(2k(\kappa_2 + \kappa_4 k^2)) - \cos(2kt) \operatorname{cosech}(2k(\kappa_2 + \kappa_4 k^2))] \quad (91)$$

The momentum integral cannot be done explicitly. But we still can plot the entanglement entropy numerically. Figure (2) are the plots of EE growth with ‘small’ and ‘large’ values of κ_4 . As expected, the entanglement entropy reaches an equilibrium quickly.

⁵We would like to thank Justin David for informing us that this reformionization could be done in principle using U(1) currents and it has not been done anywhere.

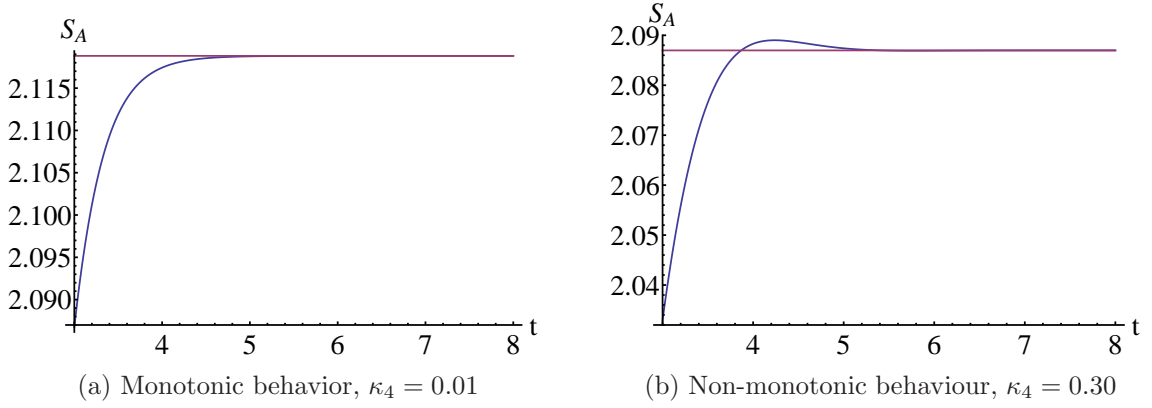


Figure 2: Entanglement entropy growth of an interval($r=5$) for different choice of κ_4 and $\kappa_2 = 1$.

The most interesting aspect of Figure (2) is the non-monotonic growth of EE in the gCC state with ‘large’ κ_4 . As in case of CC state, to study the monotonic or non-monotonic behaviour of S_A , the more appropriate quantity is not S_A but rather $\frac{\partial S_A}{\partial t}$, the expression also simplifies tremendously.

$$\begin{aligned} \left\langle \frac{\partial S_A}{\partial t} \right\rangle_{gCC} &= \frac{1}{3} \int_{-\infty}^{\infty} dk (1 - e^{ikr}) \operatorname{cosech}(2\kappa_2 k + 2\kappa_4 k^3) \sin(2kt) \\ &= \frac{1}{3} \int_{-\infty}^{\infty} dk (1 - \cos(kr)) \operatorname{cosech}(2\kappa_2 k + 2\kappa_4 k^3) \sin(2kt) \end{aligned} \quad (92)$$

Unfortunately, the above integral still cannot be done in closed form. The objective is to find finite positive real zeroes of the above expression as a function of time t . But, calculating zeroes of Fourier transforms, unless it can be done in closed form, is notoriously hard, the most famous example being the Riemann hypothesis.

The most interesting question that can be asked in Figure (2) is whether even a small infinitesimal κ_4 , although not visible in the numerical plot, gives rise to the non-monotonic EE growth or whether the non-monotonic behaviour starts from a sharp finite value of κ_4 . If it is the second case, then it is a dynamical phase transition. In other words, the question is whether (92) has finite zeroes as a function of time even for an infinitesimal κ_4 or do the finite zeroes appear for κ_4 greater than a critical value.

We found that the non-monotonic behaviour starts abruptly at a critical value of $\kappa_4 = 16\kappa_2^3/27\pi^2$, i.e., it is a dynamical phase transition. In terms of the effective temperature and chemical potential in the stationary limit, $\beta = 4\kappa_2$ and $\mu_4 = 4\kappa_4$, the critical value is $\mu_4 = \beta^3/27\pi^2$.

Although the integral (92) cannot be done in closed form, we can take advantage of the fact that for our question we do not need to know the precise zeroes. Using contour integration, the integral is given by the sum of residues of the poles given by $2\kappa_2 k + 2\kappa_4 k^3 = in\pi$ where $n \in \mathbb{Z} - \{0\}$. $n = 0$ is not a pole of (92). The expressions of the poles (from MPS)⁶

⁶The numerical values of the poles may get interchanged for specific values of the parameters but the

are

$$k_1 = \frac{-2 \cdot 6^{2/3} \kappa_2 + \sqrt[3]{6} \left(\sqrt{48\kappa_2^3 - 81\pi^2 \kappa_4 n^2} + 9i\pi \sqrt{\kappa_4} n \right)^{2/3}}{6 \sqrt[3]{\sqrt{3} \sqrt{\kappa_4^3 (16\kappa_2^3 - 27\pi^2 \kappa_4 n^2)} + 9i\pi \kappa_4^2 n}} \quad (93)$$

$$k_2 = \frac{4 \sqrt[3]{-6} \kappa_2 + i (\sqrt{3} + i) \left(\sqrt{48\kappa_2^3 - 81\pi^2 \kappa_4 n^2} + 9i\pi \sqrt{\kappa_4} n \right)^{2/3}}{2 \cdot 6^{2/3} \sqrt[3]{\sqrt{3} \sqrt{\kappa_4^3 (16\kappa_2^3 - 27\pi^2 \kappa_4 n^2)} + 9i\pi \kappa_4^2 n}} \quad (94)$$

$$k_3 = -\frac{\sqrt[3]{-1} \left(2 \sqrt[3]{-6} \kappa_2 + \left(\sqrt{48\kappa_2^3 - 81\pi^2 \kappa_4 n^2} + 9i\pi \sqrt{\kappa_4} n \right)^{2/3} \right)}{6^{2/3} \sqrt{\kappa_4} \sqrt[3]{\sqrt{48\kappa_2^3 - 81\pi^2 \kappa_4 n^2} + 9i\pi \sqrt{\kappa_4} n}} \quad (95)$$

Out of the three poles, only one is perturbative. In $\kappa_4 \rightarrow 0$ series expansion, the other two start with $\mathcal{O}(\frac{1}{\sqrt{\kappa_4}})$. One of the three poles is always imaginary for arbitrary n and arbitrary positive κ_4 .

There are three important ingredients for the proof of the dynamical phase transition:

1. All three n^{th} poles become purely imaginary when $16\kappa_2^3 - 27\pi^2 \kappa_4 n^2$ is negative, or κ_4 is greater than $16\kappa_2^3/27\pi^2 n^2$, we will call this the n^{th} critical value $\kappa_{4c,n}$,

$$\kappa_{4c,n} = \frac{16\kappa_2^3}{27\pi^2 n^2} \quad (96)$$

Below this value, the residues of the n^{th} poles are exponential decaying functions of time t , with no oscillatory factor. Obviously, $(n = \pm 1)$ critical⁷ value κ_{4c} is larger than $\kappa_{4c,n}$ for $|n| > 1$. With κ scaled to 1, κ_{4c} is $16\kappa_2^3/27\pi^2 \sim 0.0600422$.

2. With κ_4 less than $(n = \pm 1)$ critical value, the sum of the residues of $(n = \pm 1)$ poles is larger than the sum of the residues of all the other $(|n| > 1)$ poles. Hence, the behaviour of the first poles of $n = \pm 1$ dictate the behaviour of the integral (92) when $\kappa_4 < 16\kappa_2^3/27\pi^2$.
3. Above this critical value, for each n , two of the poles have real parts while one of them, say k_1 , is imaginary. The poles are

$$k_1 = -2i \operatorname{sgn}(n) b, \quad k_2 = a + i \operatorname{sgn}(n) b, \quad k_3 = -a + i \operatorname{sgn}(n) b \quad (97)$$

$$a = \frac{B^{2/3} - 2 \sqrt[3]{6} \kappa_2}{2 \cdot 2^{2/3} \sqrt[3]{3} \sqrt[3]{B} \sqrt{\kappa_4}}, \quad b = \frac{B^{2/3} + 2 \sqrt[3]{6} \kappa_2}{2 \cdot 6^{2/3} \sqrt[3]{B} \sqrt{\kappa_4}}$$

$$B = \sqrt{81\pi^2 \kappa_4 n^2 - 48\kappa_2^3 + 9\pi |n| \sqrt{\kappa_4}}$$

result will always be the same set of roots. This arises from the particular method used for solving the cubic equation.

⁷We will call this value just ‘critical value’ without the ‘ n^{th} ’ specification because, as shown below, this is the critical value of κ_4 where the dynamical phase transition happens.

where we have to take the real roots of the radicals. k_1 's have the largest imaginary parts and the exponential decay of their residues as a function of time are faster while the other poles k_2 and k_3 have comparatively large magnitudes and oscillations.⁸ In the total integral, the contributions of the imaginary poles k_1 's cannot compete with the contributions of the oscillating poles. Lastly, it would be a very special arrangement if all oscillating terms conspire to give a non-oscillatory sum. Hence, the total integral is oscillatory as a function of time and the EE growth is non-monotonic.

For future reference, we also note that the expansion of the real part 'a' in (97) around the n^{th} critical value $\kappa_{4c,n}$ is

$$a = \frac{\sqrt[3]{\pi} \sqrt[3]{|n|} \sqrt{\kappa_4 - \kappa_{4c,n}}}{2^{2/3} \sqrt{3} \kappa_{4c,n}^{5/6}} - \frac{35 \left(\sqrt[3]{\pi} \sqrt[3]{|n|} \right) (\kappa_4 - \kappa_{4c,n})^{3/2}}{54 \left(2^{2/3} \sqrt{3} \kappa_{4c,n}^{11/6} \right)} + \frac{1001 \sqrt[3]{\pi} \sqrt[3]{|n|} (\kappa_4 - \kappa_{4c,n})^{5/2}}{1944 \cdot 2^{2/3} \sqrt{3} \kappa_{4c,n}^{17/6}} + \mathcal{O}(\kappa_4 - \kappa_{4c,n})^{7/2} \quad (98)$$

For all our calculations below, we have scaled κ_2 to be 1. The first point is clear from figure (3). The real parts of ($n = \pm 1$) poles vanish at $\kappa_4 \sim 0.060$, which is the critical value found above. The critical value of ($n = \pm 2$) poles is $\kappa_4 \sim 0.015$.

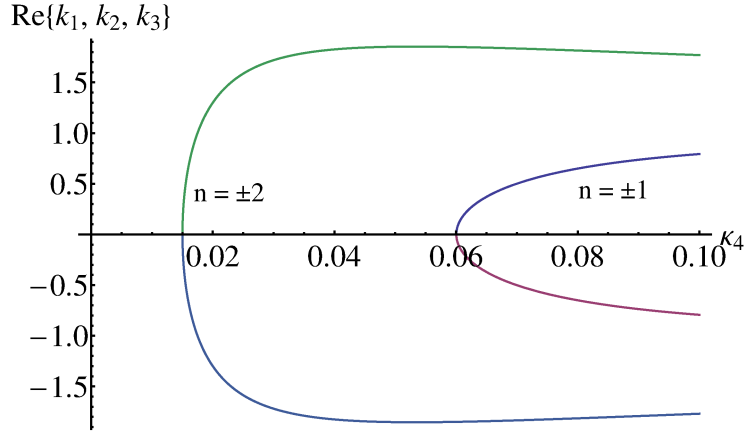


Figure 3: Real parts of poles of $n \in \{\pm 1, \pm 2\}$ as a function of κ_4 with κ_2 scaled to 1.

Below the critical value, we will show that the total contributions from $n = \pm 1$ poles is larger than the sum of all residues of $|n| > 1$ poles. We will concentrate on the late time period, $t > r/2$. For e^{i2kt} of $\sin(2kt)$ factor in (92), the contour is closed upward encircling the upper half plane, and for e^{-i2kt} , the contour is closed downward encircling the lower half plane. From the expansion of $\text{cosech}(2\kappa_4(k - k_1)(k - k_2)(k - k_3) + i\pi n)$ around k_1 ,

⁸This competition between poles of each n might be important, if we have turned on W_6 chemical potential instead of W_4 , in which case there will be five poles, or W_8 in which case there will be seven poles and so on.

the contribution from k_1 poles for arbitrary n are the real parts of

$$P_n(k_1) = 2\pi i R_1(k_1) = \frac{(-1)^n}{6\kappa_4(k_1 - k_2)(k_1 - k_3)} \left(e^{i2k_1 t} - \frac{e^{ik_1(r+2t)} + e^{ik_1(-r+2t)}}{2} \right) \quad \text{if } \text{Im}[k_1] > 0 \quad (99)$$

$$Q_n(k_1) = -2\pi i R_2(k_1) = \frac{(-1)^n}{6\kappa_4(k_1 - k_2)(k_1 - k_3)} \left(e^{-i2k_1 t} - \frac{e^{ik_1(r-2t)} + e^{-ik_1(r+2t)}}{2} \right) \quad \text{if } \text{Im}[k_1] < 0 \quad (100)$$

where R_1 and R_2 denote the residues. Similarly, cyclic replacements of k_1 with k_2 and k_3 give the contributions of k_2 and k_3 poles. For the poles in the lower half of the complex plane, since the contour is anticlockwise, Q_n have an extra minus sign in the residue. We will call the contributions to the integral form $n = \pm 1$ poles as $I_0(t)$ and the contributions of the $|n| > 1$ poles as $I_1(t)$. The other parameters (κ_4 , r and κ which is already scaled to 1) are suppressed.

As a first visual evidence, Figure (4) is the comparison of numerical integration of (92) and $I_0(t)$. It is evident that the residues of ($n = \pm 1$) poles dominate the contour integration. We have chosen $\kappa_4 = 0.0600420$ which is very close to the critical value. As mentioned above, with this choice, all the poles except the $n = \pm 1$ poles give oscillating residues as a function of time. Although it is not very conspicuous, it is also evident from the graph that $I_1(t)$ is oscillating around $I_0(t)$, the value of the numerical integration is above the $I_0(t)$ curve in some regions and below in other regions of time t .

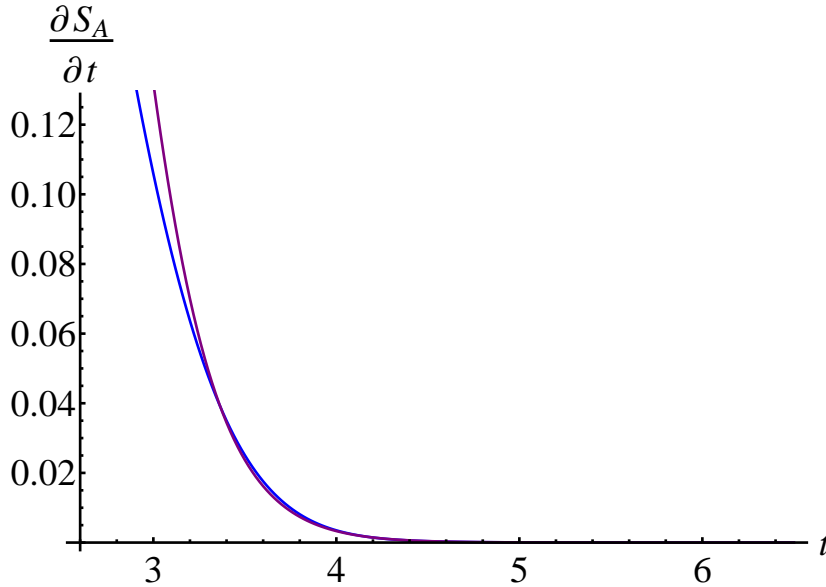


Figure 4: Comparison of numerical integration of $\langle \partial S_A / \partial t \rangle_{g_{CC}}$ (blue curve) and $I_0(t)$ (purple curve) as a function of time t . The parameters are $\kappa_4 = 0.0600420$, $r = 5$.

The numerical integration is unreliable in the long time limit. So, to complete our argument, we will calculate an upper bound of $I_1(t)$ and compare it with $I_0(t)$ for a specific

time t . The choice of the parameters are

$$\kappa = 1, \kappa_4 = 0.0600420, r = 5, t = 4r = 20, \quad (101)$$

With these parameters, the $n = 1$ and $n = -1$ poles are

$$k_1 = 2.3538234i \quad k_2 = 2.3585719i \quad k_3 = -4.7123954i \quad ; n = 1 \quad (102)$$

$$k_1 = 4.7123954i \quad k_2 = -2.3538234i \quad k_3 = -2.3585719i \quad ; n = -1 \quad (103)$$

and $I_0(t)$ is given by

$$\begin{aligned} I_0(t)|_{t=20} &= P(k_1)|_{n=1} + P(k_2)|_{n=1} + Q(k_3)|_{n=1} + P(k_1)|_{n=-1} + Q(k_2)|_{n=-1} + Q(k_3)|_{n=-1} \\ &= 6.646589 \times 10^{-35} \end{aligned} \quad (104)$$

We can show that $I_1(t)|_{t=20}$ is less than $I_0(t)|_{t=20}$. The first few poles are

$$\begin{aligned} k_1 &= 5.5495551i & k_2 &= 2.5383386 - 2.7747775i & k_3 &= -2.5383386 - 2.7747775i & ; n &= -3 \\ k_1 &= 5.1737935i, & k_2 &= 1.8496206 - 2.5868967i & k_3 &= -1.8496206 - 2.5868967i & ; n &= -2 \\ k_1 &= -5.1737935i & k_2 &= -1.8496206 + 2.5868967i & k_3 &= 1.8496206 + 2.5868967i & ; n &= 2 \\ k_1 &= -5.54955505i & k_2 &= -2.5383386 + 2.7747775i & k_3 &= 2.5383386 + 2.7747775i & ; n &= 3 \end{aligned}$$

The residues of these ($|n| > 1$) poles cannot be summed up into a closed form, as that would amount to doing the integral in closed form. We are interested in an upper bound. The residues of two of the three poles of every ($|n| > 1$) have an oscillation factor. As we saw, even each residue has a separate 3-6 real oscillating terms as a function of time. So, we can represent the sum of the modulus (absolute value of the amplitude) of the oscillating terms of the three residues for each n , by a bigger function which has the analytic sum from $|n| > 1$ to infinity. And if the sum is less $I_0(t)$, then $I_0(t)$ dominates the contribution from all the other poles.⁹

⁹A simplified example of our strategy is the comparison between say X and $a \sin(x) + b \cos(y)$ where $\{X, a, b, x, y\} \in \mathcal{R}$, while $A > |a|$ and $B > |b|$ and $\{A, B\} \in \mathcal{R}^+$, then $A + B > |a| + |b| > a \sin(x) + b \cos(y)$ and if $X > A + B$ then $X > a \sin(x) + b \cos(y)$.

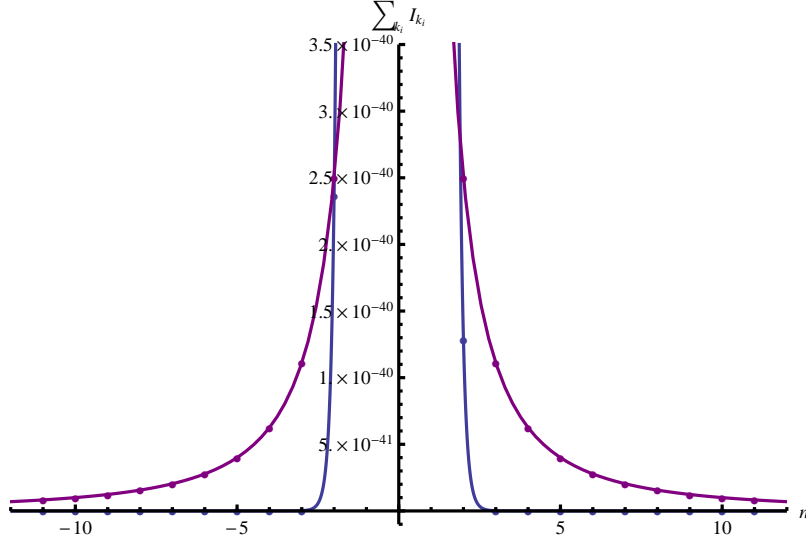


Figure 5: Comparison of sum of modulus of residues of ($|n| > 1$) poles with the approximating function $f(n) = 10^{-39}/n^2$. The dots are the discrete n values of the corresponding functions.

Figure (5) are the plots of the sum of the moduli separately for the oscillating terms of the three residues as a function of n and the approximating function $f(n) = 10^{-39}/n^2$. Now, we have

$$\sum_{n=-\infty}^{n=-2} \frac{10^{-39}}{n^2} + \sum_{n=2}^{\infty} \frac{10^{-39}}{n^2} = 1.289868 \times 10^{-39} \quad (105)$$

This is much less than $I_0(t)|_{t=20}$ in (104) and is of the order of 10^{-5} of $I_0(t)|_{t=20}$. So, the non-oscillating $I_0(t)$ dominates $I_1(t)$, the contribution from the other poles. Hence, below $\kappa_4 = 16\kappa_2^3/27\pi^2$, the EE growth is monotonic.

Visually from figure (4), $t = 3.7$ is a time-slice where the difference between $I_0(t)$ and the numerical integration has a local maxima. At this time slice, repeating the above exercise, $I_0(t)|_{t=3.7} = 0.109727$ and repeating the same exercise of estimating the upper bound of $I_1(t)|_{t=3.7}$ with the same parameters as (101) except the change in t , we get a good upper bound to be 0.0064493 which is less than $I_0(t)|_{t=3.7}$ and is of the order of 60% of $I_0(t)|_{t=3.7}$. So, the approximation of the full integral by $I_0(t)$ gets better with increasing time. In the long time limit, we can effectively take the only time-dependence to be the time-dependence of $I_0(t)$. *It is worth mentioning here that even ($n = \pm 1$) pole calculations take into account κ_4 non-perturbatively, since two of the poles of each n are non-perturbative in κ_4 .*

As listed above as one of the main points, above the critical value, each n has an imaginary pole but the other two poles have real parts and also have larger magnitudes so the total residue of the three poles of each n is oscillatory. It would also be a very special arrangement if all the oscillatory contributions of each n conspire to give a non-oscillatory $\partial S_A/\partial t$. Hence, we conclude that the EE growth is non-monotonic above the critical value.

Near the critical point $(\kappa_4 - \kappa_{4c}) \rightarrow 0^+$, we can try to estimate an upper bound of the time upto which the EE growth is monotonic. The upper bound is half of the longest time period. Using the leading term in expansion of ‘ a ’ from (98) and the expressions of the

residues (99) and (100), the lowest frequency ($|n| = 1$) gives the upper bound as

$$\frac{\sqrt[3]{\pi}\sqrt{\kappa_4 - \kappa_{4c}}}{2^{2/3}\sqrt{3}\kappa_{4c}^{5/6}}(2t - r) = \pi \quad \Rightarrow \quad t = \frac{(2\pi)^{2/3}\sqrt{3}\kappa_{4c}^{5/6}}{2\sqrt{\kappa_4 - \kappa_{4c}}} + \frac{r}{2} \sim \frac{2.95 \kappa_{4c}^{5/6}}{\sqrt{\kappa_4 - \kappa_{4c}}} \quad (106)$$

where finite ‘ r ’ can be neglected in the limit $(\kappa_4 - \kappa_{4c}) \rightarrow 0^+$.

The critical value in terms the effective temperature $\beta = 4\kappa_2$ and chemical potential $\mu_4 = 4\kappa^4$ in the stationary limit is

$$\mu_4 = \frac{\beta^3}{27\pi^2} \quad (107)$$

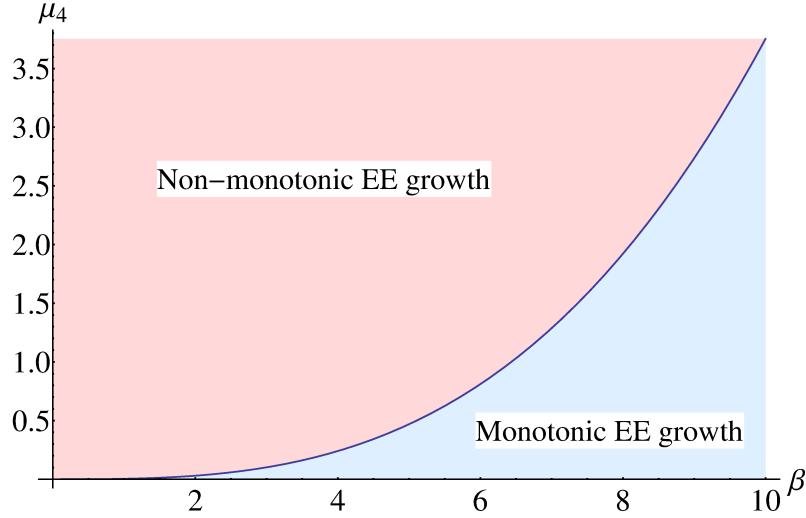


Figure 6: The critical curve $\mu_4 = \beta^3/27\pi^2$ in terms of the effective temperature and chemical potential in the stationary limit and the phase diagram.

For the early times $t < r/2$, in the residue calculations (99) and (100), we have to replace the sign of the exponents with $r - 2t$ so the magnitudes of the exponentials decreases as time increases. Upto the critical value of κ_4 , the EE growth is always monotonic for this time period.

7.1 Turning on other charges

We could also calculate the EE growth of gCC states with other charges of the fermionic theory corresponding to bosonic bilinear $W_{2n} = \sum |k|^{2n-1} d_k^\dagger d_k$ where $n = 3, 4, 5, \dots$. Repeating the exercise of quenching tuned squeezed states of scalar field theory in MPS, the propagator with these charges are simply given by

$$\langle \phi(0, t) \phi(r, t) \rangle = \int \frac{dk}{4\pi} \frac{e^{ikr}}{k} \left(\coth \left(2\kappa_2 k + \sum_{n=2}^{\infty} \kappa_{2n} k^{2n-1} \right) - \cos(2kt) \operatorname{cosech} \left(2\kappa_2 k + \sum_{n=2}^{\infty} \kappa_{2n} k^{2n-1} \right) - 1 \right) \quad (108)$$

Substituting this propagator in the general formula (77) and (78) give the Rényi entropy and entanglement entropy. The first derivative of EE w.r.t. time is

$$\left\langle \frac{\partial S_A}{\partial t} \right\rangle_{gCC} = \frac{1}{3} \int_{-\infty}^{\infty} dk (1 - \cos(kr)) \text{cosech} \left(2\kappa k + \sum_{n=2}^{\infty} \kappa_{2n} k^{2n-1} \right) \sin(2kt) \quad (109)$$

We believe the dynamics will be much richer with these other charges, with much more complex phase diagrams which can be in a $n - 1$ dimensional space. But the general poles analysis cannot be done in these cases because the poles will be given by quintic and higher order equations. Considering gCC states with W_4 and W_6 charges, the numerical plots of EE growth looks the same as (2) where by trial and error method, some parameter subspace gives monotonic growth and some subspaces do not give monotonic growth. Considering $n = \pm 1$, the poles are given by $2\kappa_2 k + 2\kappa_4 k^3 + 2\kappa_6 k^5 = i\pi$. For $\kappa_2 = 1$ and $\kappa_4 = 0.06$, numerically we find two interesting parameter subspaces of κ_6 . The first one is when all the poles become imaginary when κ_4 is decreased.

$$\begin{aligned} k_1 &= 2.0887597 \operatorname{sgn}(n) i, & k_2 &= 2.9527785 \operatorname{sgn}(n) i, & k_3 &= -6.5425830 \operatorname{sgn}(n) i, \\ k_4 &= -6.6158300 \operatorname{sgn}(n) i, & k_5 &= 8.1168748 \operatorname{sgn}(n) i & \text{for } \kappa_6 &= 0.0007249 \end{aligned} \quad (110)$$

$$\begin{aligned} k_1 &= -0.0076887 - 6.5788763 \operatorname{sgn}(n) i, & k_2 &= 2.0887456 \operatorname{sgn}(n) i, & k_3 &= 2.9528549 \operatorname{sgn}(n) i, \\ k_4 &= 8.1161520 \operatorname{sgn}(n) i, & k_5 &= 0.0076887 - 6.5788763 \operatorname{sgn}(n) i & \text{for } \kappa_6 &= 0.0007250 \end{aligned} \quad (111)$$

This looks like the same transition if $n = \pm 1$ dominates, but the poles with real parts have large imaginary part also, so they would be highly damped. The other case is

$$\begin{aligned} k_1 &= -0.8215058 + 1.9681831 \operatorname{sgn}(n) i, & k_2 &= -5.2389645 \operatorname{sgn}(n) i, & k_3 &= -5.2472000 \operatorname{sgn}(n) i, \\ k_4 &= 6.5497983 \operatorname{sgn}(n) i, & k_5 &= 0.8215058 + 1.9681831 \operatorname{sgn}(n) i & \text{for } \kappa_6 &= 0.0019179 \end{aligned} \quad (112)$$

$$\begin{aligned} k_1 &= -0.8215060 + 1.9681836 \operatorname{sgn}(n) i, & k_2 &= -0.0040372 - 5.2430724 \operatorname{sgn}(n) i, \\ k_3 &= 6.5497775 \operatorname{sgn}(n) i, & k_4 &= 0.0040372 - 5.2430724 \operatorname{sgn}(n) i, \\ k_5 &= 0.8215060 + 1.9681836 \operatorname{sgn}(n) i & \text{for } \kappa_6 &= 0.0019180 \end{aligned} \quad (113)$$

for the smaller κ_4 , although two of the poles have real parts, they have to compete with the three imaginary poles. So, this could also be phase transition.

8 Discussion

In this paper, we have examined free fermionic mass quench. We find that the ground state quench equilibrates but not to a thermal ensemble. Starting from specially prepared squeezed states, we get CC and gCC states with fermionic bilinear W_{2n} charges. Calculation of correlators in CC and gCC states explicitly shows thermalization to thermal ensemble and GGE respectively.

For CC state, we calculate EE growth exactly. The EE growth is strictly monotonically increasing. For gCC state with a particular charge, we find dynamical phase transition in which the EE growth is monotonic upto a critical value of the effective chemical potential. In the pure state, the effective chemical potential is the coupling constant of the current corresponding to the charge. Above the critical value, the EE growth is non-monotonic. It would be interesting to reproduce our result in large c holographic CFTs and examine what it would mean for Black hole physics.

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A Conventions

$$\begin{aligned}\eta_{\mu\nu} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \partial_\mu = (\partial_t, \partial_x), \quad \gamma^\mu \partial_\mu = \gamma^0 \partial_t - \gamma^1 \partial_x, \\ w &= t - x, \quad \bar{w} = t + x, \quad \partial = \frac{\partial}{\partial w} = \frac{1}{2} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right), \quad \bar{\partial} = \frac{\partial}{\partial \bar{w}} = \frac{1}{2} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \\ \gamma_d^0 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma_d^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{in Dirac basis.} \\ S &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \gamma_c^0 = S \gamma_d^0 S^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_c^1 = S \gamma_d^1 S^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{in chiral basis.} \\ \{a_k, a_{k'}^\dagger\} &= 2\pi \delta(k - k'), \quad \{b_k, b_{k'}^\dagger\} = 2\pi \delta(k - k'), \quad \text{other anticommutators are zero.} \\ \{a_n, a_{n'}^\dagger\} &= \delta(n - n'), \quad \{b_n, b_{n'}^\dagger\} = \delta(n - n'), \quad \text{other anticommutators are zero.}\end{aligned}$$

We will use $k = \frac{2\pi n}{L}$ for continuum limit ($L \rightarrow \infty$) and n for quantization in a finite box of size L , where $n = n' + 1/2$ and $n' \in \mathbb{Z}$.

B Spinors and transformation to chiral basis:

Taking constant mass m , we can easily find the boosted spinors, $u(k, m)$ and $v(k, m)$. For constant m , $\phi_{+p}(t) = e^{-i\omega t}$ and $\phi_{-m}(t) = e^{i\omega t}$. So from (4),

$$\begin{aligned}U(x, t) &= [\gamma^0 \partial_t - \gamma^1 \partial_x - im] e^{-i\omega t + ikx} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i \begin{bmatrix} -(\omega + m) \\ k \end{bmatrix} e^{-ik \cdot x} \\ V(x, t) &= [\gamma^0 \partial_t - \gamma^1 \partial_x - im] \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{ik \cdot x} = i \begin{bmatrix} k \\ -(\omega + m) \end{bmatrix} e^{ik \cdot x}\end{aligned}$$

Hence, upto normalizations fixed by inner products, the boosted spinors are

$$u(k, m) = i \begin{bmatrix} -(\omega + m) \\ k \end{bmatrix}, \quad v(k, m) = i \begin{bmatrix} k \\ -(\omega + m) \end{bmatrix}$$

We have the adjoint spinors as,

$$\begin{aligned} \bar{u}(k, m) &= u^\dagger(k, m)\gamma^0 = -i \begin{bmatrix} -(\omega + m) & k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i \begin{bmatrix} -(\omega + m) & -k \end{bmatrix} \\ \bar{v}(k, m) &= v^\dagger(k, m)\gamma^0 = -i \begin{bmatrix} k & -(\omega + m) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i \begin{bmatrix} k & (\omega + m) \end{bmatrix} \end{aligned}$$

Now borrowing Peskin & Schroeder(P&S) conventions of spinors, we want to fix the inner products $\bar{u}(k, m)u(k, m) = 2m$ and $\bar{v}(k, m)v(k, m) = -2m$,

$$\begin{aligned} \bar{u}(k, m)u(k, m) &= \begin{bmatrix} -(\omega + m) & -k \end{bmatrix} \begin{bmatrix} -(\omega + m) \\ k \end{bmatrix} = 2m(\omega + m) \\ \bar{v}(k, m)v(k, m) &= \begin{bmatrix} k & (\omega + m) \end{bmatrix} \begin{bmatrix} k \\ -(\omega + m) \end{bmatrix} = -2m(\omega + m) \end{aligned}$$

So the normalized spinors are

$$\begin{aligned} u(k, m) &= \frac{1}{\sqrt{(\omega + m)}} \begin{bmatrix} (\omega + m) \\ -k \end{bmatrix}, \quad v(k, m) = \frac{1}{\sqrt{(\omega + m)}} \begin{bmatrix} k \\ -(\omega + m) \end{bmatrix} \\ \bar{u}(k, m) &= \frac{1}{\sqrt{(\omega + m)}} \begin{bmatrix} (\omega + m) & k \end{bmatrix}, \quad \bar{v}(k, m) = \frac{1}{\sqrt{(\omega + m)}} \begin{bmatrix} k & (\omega + m) \end{bmatrix} \end{aligned} \tag{114}$$

The spinors with time-dependent mass $m(t)$ are obtained by just substituting $m(t)$ in the place of ‘ m ’ only inside the matrices, which is clearly seen from (12) and (13). The normalization cannot be changed to time-dependent mass else the spinors won’t be solutions of the corresponding Dirac equation.

The transformation to chiral basis is accomplished by using the transformation matrix $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The mode expansion as in P&S is

$$\begin{aligned} \Psi(x, t) &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega}} \left[a_k u(k, m) e^{-ik \cdot x} + b_k^\dagger v(k, m) e^{ik \cdot x} \right] \\ &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega}} \left[a_k \frac{1}{\sqrt{(\omega + m)}} \begin{bmatrix} (\omega + m) \\ -k \end{bmatrix} e^{-ik \cdot x} + b_k^\dagger \frac{1}{\sqrt{(\omega + m)}} \begin{bmatrix} k \\ -(\omega + m) \end{bmatrix} e^{ik \cdot x} \right] \\ &\xrightarrow{m \rightarrow 0} \int \frac{dk}{2\pi} \frac{1}{\sqrt{2}} \left[a_k \begin{bmatrix} 1 \\ -\text{sgn}(k) \end{bmatrix} e^{-ik \cdot x} + b_k^\dagger \begin{bmatrix} \text{sgn}(k) \\ -1 \end{bmatrix} e^{ik \cdot x} \right] \\ &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2}} \left[a_k e^{-ik \cdot x} + \text{sgn}(k) b_k^\dagger e^{ik \cdot x} \right. \\ &\quad \left. - \text{sgn}(k) a_k e^{-ik \cdot x} - b_k^\dagger e^{ik \cdot x} \right] \end{aligned}$$

In the chiral basis,

$$\begin{aligned}\Psi_c(x, t) &= S \cdot \Psi(x, t) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \int \frac{dk}{2\pi} \frac{1}{\sqrt{2}} \begin{bmatrix} a_k e^{-ik \cdot x} + \text{sgn}(k) b_k^\dagger e^{ik \cdot x} \\ -\text{sgn}(k) a_k e^{-ik \cdot x} - b_k^\dagger e^{ik \cdot x} \end{bmatrix} \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{2} \begin{bmatrix} (1 + \text{sgn}(k))(a_k e^{-ik \cdot x} + b_k^\dagger e^{ik \cdot x}) \\ (1 - \text{sgn}(k))(a_k e^{-ik \cdot x} - b_k^\dagger e^{ik \cdot x}) \end{bmatrix}\end{aligned}\quad (115)$$

Writing this as $\psi(x, t)$ and $\bar{\psi}(x, t)$,

$$\psi(x, t) = \int_0^\infty \frac{dk}{2\pi} (a_k e^{-ik \cdot x} + b_k^\dagger e^{ik \cdot x}) \quad (116)$$

$$\bar{\psi}(x, t) = \int_{-\infty}^0 \frac{dk}{2\pi} (a_k e^{-ik \cdot x} - b_k^\dagger e^{ik \cdot x}) \quad (117)$$

C Fermionic Boundary State

The action (1) with $m(t) = 0$ in the chiral basis is

$$\begin{aligned}S &= - \int dx^2 [i \bar{\Psi} \gamma^\mu \partial_\mu \Psi + \Psi \gamma^\mu \partial_\mu \bar{\Psi}] \\ &= -\frac{i}{2} \int dw d\bar{w} (\psi^\dagger \bar{\partial} \psi + \bar{\psi}^\dagger \partial \bar{\psi} + \psi \bar{\partial} \psi^\dagger + \bar{\psi} \partial \bar{\psi}^\dagger)\end{aligned}$$

On varying the action and collecting terms, we get the following

$$\delta S = \int d^2x (\delta \psi^\dagger \bar{\partial} \psi + \delta \psi \bar{\partial} \psi^\dagger + \delta \bar{\psi}^\dagger \partial \bar{\psi} + \delta \bar{\psi} \partial \bar{\psi}^\dagger) \quad (118)$$

Given a boundary at $t = 0$, it will also have certain boundary terms, which we want to be zero.

$$\psi^\dagger \delta \psi + \psi \delta \psi^\dagger + \bar{\psi}^\dagger \delta \bar{\psi} + \bar{\psi} \delta \bar{\psi}^\dagger \big|_{t=0} = 0 \quad (119)$$

We impose this as an operator equation on the boundary state $|B\rangle$. The condition for a boundary state can be achieved via two identifications

$$\psi = i \bar{\psi}^\dagger, \quad \text{and} \quad \psi^\dagger = i \bar{\psi} \quad (120)$$

$$\psi = -\bar{\psi}, \quad \text{and} \quad \psi^\dagger = \bar{\psi}^\dagger \quad (121)$$

Now we impose the boundary conditions at $t = 0$ in terms of the mode expansions (116) and (117):

1. The boundary condition of (120) gives $a_k \mp i a_{-k}^\dagger = 0$ for $k > 0$ and $b_k \mp i b_{-k}^\dagger = 0$ for $k < 0$. Similarly, the second condition is $a_k \pm i a_{-k}^\dagger = 0$ for $k < 0$ and $b_k \pm i b_{-k}^\dagger = 0$ for $k > 0$. Combining the separate conditions, we get $a_k \mp i \text{sgn}(k) a_{-k}^\dagger = 0$ and $b_k \pm i \text{sgn}(k) b_{-k}^\dagger = 0$. Hence, the boundary state corresponding to the first identification is

$$|N\rangle = \exp \left(\sum_k i \text{sgn}(k) (a_k^\dagger a_{-k}^\dagger - b_k^\dagger b_{-k}^\dagger) \right) |0\rangle \quad (122)$$

2. The boundary condition (121) is $a_k \mp \text{sgn}(k)b_{-k}^\dagger = 0$ for $k > 0$, $a_k \pm b_{-k}^\dagger = 0$ for $k < 0$ and $b_k \pm a_{-k}^\dagger = 0$ for $k < 0$ and $b_k \mp a_{-k}^\dagger = 0$ for $k > 0$. The boundary state for the first identification is

$$|D\rangle = \exp\left(\sum_k \text{sgn}(k)a_k^\dagger b_{-k}^\dagger\right)|0\rangle \quad (123)$$

From the action S , we can find the non-zero components of the energy-momentum tensor $T = T_{ww}$ and $\bar{T} = T_{\bar{w}\bar{w}}$, and the components of the $U(1)$ current are $J_w = J$ and $J_{\bar{w}} = \bar{J}$,

$$T = \frac{i}{2}(\psi^\dagger \partial \psi + \psi \partial \psi^\dagger) \quad \bar{T} = \frac{i}{2}(\bar{\psi}^\dagger \partial \bar{\psi} + \bar{\psi} \partial \bar{\psi}^\dagger) \quad (124)$$

$$J = \psi^\dagger \psi \quad \bar{J} = \bar{\psi}^\dagger \bar{\psi} \quad (125)$$

The boundary conditions (120) and (121) satisfy the condition

$$T(w)|_{t=0} = \bar{T}(\bar{w})|_{t=0}, \quad \text{on the cylinder.} \quad (126)$$

$$\text{or, } (z^2 T_{zz}(z))|_{z(t=0)} = (\bar{z}^2 \bar{T}_{\bar{z}\bar{z}}(\bar{z}))|_{\bar{z}(t=0)}, \quad \text{on the plane.} \quad (127)$$

where $z = e^{2\pi(t-ix)/L}$ and $\bar{z} = e^{2\pi(t+ix)/L}$. Thus $|N\rangle$ and $|D\rangle$ are conformal invariant boundary states. It is also worth noting that the boundary conditions also satisfy

$$J(w)|_{t=0} = -\bar{J}(\bar{w})|_{t=0}, \quad \text{on the cylinder.} \quad (128)$$

$$\text{or, } (z J_z(z))|_{z(t=0)} = (\bar{z} \bar{J}_{\bar{z}}(\bar{z}))|_{\bar{z}(t=0)}, \quad \text{on the plane.} \quad (129)$$

Considering the zero modes in the cylinder, it means that the above boundary states are not charged. With $Q = J_0 + \bar{J}_0$,

$$Q|N\rangle = 0, \quad Q|D\rangle = 0 \quad (130)$$

Besides, specially for the state $|D\rangle$, $(J_0 - \bar{J}_0)|D\rangle = 0$. Hence

$$J_0|D\rangle = 0 \quad \bar{J}_0|D\rangle = 0 \quad (131)$$

D Bosonised Boundary State

Consider a Dirichlet boundary state $\varphi|D\rangle = -\bar{\varphi}|D\rangle$. Using the bosonised fermions :

$$\begin{aligned} \psi &= e^{-i\frac{\sqrt{\pi}}{4}\bar{P}} : e^{-i\sqrt{4\pi}\varphi(w)} : & \psi^\dagger &= e^{i\frac{\sqrt{\pi}}{4}\bar{P}} : e^{i\sqrt{4\pi}\varphi(w)} : \\ \bar{\psi} &= e^{-i\frac{\sqrt{\pi}}{4}P} : e^{i\sqrt{4\pi}\bar{\varphi}(\bar{w})} : & \bar{\psi}^\dagger &= e^{i\frac{\sqrt{\pi}}{4}P} : e^{-i\sqrt{4\pi}\bar{\varphi}(\bar{w})} : \end{aligned}$$

To translate the boson Dirichlet condition into the fermionic one, we get

$$\begin{aligned} \psi|D\rangle &= e^{-i\frac{\sqrt{\pi}}{4}\bar{P}} : e^{-i\sqrt{4\pi}\varphi} : |D\rangle \\ &= e^{-\frac{\pi}{2}[Q,P]} e^{-i\frac{\sqrt{\pi}}{4}\bar{P}} e^{-i\frac{\sqrt{\pi}}{4}P} : e^{-i\sqrt{4\pi}\varphi} : |D\rangle \\ &= e^{-\frac{\pi}{2}[Q,P]} e^{-i\frac{\sqrt{\pi}}{4}\bar{P}} e^{-i\frac{\sqrt{\pi}}{4}P} : e^{i\sqrt{4\pi}\bar{\varphi}} : |D\rangle \\ &= e^{\frac{\pi}{2}([Q,P] - [\bar{P},\bar{Q}])} \bar{\psi}|D\rangle \\ &= e^{-i\pi} \bar{\psi}|D\rangle = -\bar{\psi}|D\rangle \end{aligned}$$

where we have used the relation $e^x e^y = e^{y+[x,y]+\dots} e^x$ and $[Q, P] = [\bar{Q}, \bar{P}] = i$. We have also used (131) which gives $P|D\rangle = J_0|D\rangle = 0$ and $\bar{P}|D\rangle = \bar{J}_0|D\rangle = 0$.

Similarly, we can show that $(\psi^\dagger - \bar{\psi}^\dagger)|D\rangle$, $(\psi - i\bar{\psi}^\dagger)|N\rangle$ and $(\psi^\dagger - i\bar{\psi})|N\rangle$ vanish, where $|N\rangle$ is defined by $(\varphi - \bar{\varphi})|N\rangle = 0$ which is the Neumann boundary condition for scalar fields.

E Baker-Campbell-Hausdorff(BCH) formula

Although we are interested in the ‘out’ massless oscillators, the BCH formula is valid for both massive and massless oscillators. So, we will suppress the ‘in’ or ‘out’ identification of the oscillators. Starting from

$$|\Psi\rangle = \exp\left(\sum_k \text{sgn}(k)\gamma(k)a_k^\dagger b_{-k}^\dagger\right)|0\rangle \quad (132)$$

we wish to obtain an expression of the form

$$|\psi\rangle = \exp\left(-\sum_k \kappa(k)(a_k^\dagger a_k + b_k^\dagger b_k)\right) \exp\left(\sum_k \text{sgn}(k)a_k^\dagger b_{-k}^\dagger\right)|0\rangle \quad (133)$$

Commuting $\exp\left(-\sum_k \kappa(k)(a_k^\dagger a_k + b_k^\dagger b_k)\right)$ through $\exp\left(\sum_k \text{sgn}(k)a_k^\dagger b_{-k}^\dagger\right)$, we get

$$\exp\left(\sum_k \text{sgn}(k)e^{-2\kappa(k)}a_k^\dagger b_{-k}^\dagger\right)|0\rangle$$

Thus,

$$\begin{aligned} \text{sgn}(k)\gamma(k) &= e^{-2\kappa(k)} \text{sgn}(k) \\ \Rightarrow \kappa(k) &= -\frac{1}{2} \log(\gamma(k)) \end{aligned} \quad (134)$$

F Refermionization of bosonic bilinear \mathcal{W}_4

The bosonic(real scalar) bilinear \mathcal{W}_4 current [9, 10] is

$$\mathcal{W}_4(w) = 2\partial\phi\partial^3\phi - 3\partial^2\phi\partial^2\phi \quad (135)$$

Using U(1) current relation $J = \psi^\dagger\psi = \frac{i}{\sqrt{4\pi}}\partial\phi$ and normal ordering gives the refermionized \mathcal{W}_4 current. Because of the fermionic anti-commutation relation most of the four fermion terms drop out and the only four fermion term that survives is $\partial\psi^\dagger\partial\psi\psi^\dagger\psi$. Finally, the expression is

$$\tilde{\mathcal{W}}_4(w) = \frac{7i}{6}\psi^\dagger\partial^3\psi + \frac{3i}{2}\partial^2\psi^\dagger\partial\psi - \frac{3i}{2}\partial\psi^\dagger\partial^2\psi - \frac{7i}{6}\partial^3\psi^\dagger\psi - 2\partial\psi^\dagger\partial\psi\psi^\dagger\psi \quad (136)$$

And the corresponding charge is

$$\begin{aligned} \tilde{W}_4 = & \frac{1}{4\pi} \left(\frac{14}{3} \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k|^3 \left[a_k^\dagger a_k + b_k^\dagger b_k \right] \right. \\ & + 2 \int_{-\infty}^{\infty} \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^4} |k_1| |k_2| \left[a_{k_1}^\dagger a_{k_2} a_{k_3}^\dagger a_{k_4} \delta(k_1 - k_2 + k_3 - k_4) + a_{k_1}^\dagger a_{k_2} b_{k_3}^\dagger b_{k_4} \delta(k_1 - k_2 - k_3 + k_4) \right. \\ & - a_{k_1}^\dagger b_{k_2}^\dagger b_{k_3} a_{k_4} \delta(k_1 + k_2 - k_3 - k_4) - b_{k_1} a_{k_2} a_{k_3}^\dagger b_{k_4}^\dagger \delta(-k_1 - k_2 + k_3 + k_4) \\ & \left. \left. + b_{k_1} b_{k_2}^\dagger a_{k_3}^\dagger a_{k_4} \delta(-k_1 + k_2 + k_3 - k_4) + b_{k_1} b_{k_2}^\dagger b_{k_3}^\dagger b_{k_4} \delta(-k_1 + k_2 - k_3 + k_4) \right] \right) \end{aligned} \quad (137)$$

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